CHAPTER 2 PROBABILITIES

I Fundamental principle of enumeration

Introduction

* **Question:** How much code can be formed using a vowel followed by a number among the 1,2 and 3?

* **Answer:** There are 6 vowels a, e, i, o, u and y, so to choose a vowel, you have 6 possibilities.
For each vowel there are 3 possibilities to choose from 1, 2 and 3.
So the number of possible codes is: $N = 6 \times 3 = 18$.

Property

If a counting situation requires p choices $C_1, C_2, C_3, \ldots, C_p$ and if:

▶ The choice C_1 can produce n_1 possible results;
▶ The choice C_2 can produce n_2 possible results;
▶ The choice C_3 can produce n_3 possible results;
▶ and so on;

Then the enumeration situation can be done from $n_1 \times n_2 \times n_3 \times \ldots \times n_p$ different ways.

Example

1. Determine the number of 3-letter words chosen from 26 letters (meaningful or not).
   To make one, we choose successively:
   ◆ A first letter in the alphabet: 26 choices.
   ◆ A second letter in the alphabet: 26 choices.
   ◆ A third letter in the alphabet: 26 choices.
   There are therefore $26^3 = 17576$ distinct words of 3 letters.
2. A telephone network call number is 8 digits long. It follows that the capacity of this telephone network is: 10^8.

Application

1. How many ways can you park 4 cars in 4 empty spaces in a car park?
2. A die is rolled 2 times in a row. What are the possible outcomes?
3. Consider the following numbers: 5 – 6 – 7 – 2 – 3.
   We want to form the natural numbers of three distinct digits two by two among these digits.
   a- What is the number of possible cases?
   b- How many even numbers can be formed?
   c- How many multiples of 5 can be formed?

II Arrangement and combination

1. Arrangement with repetition

Introduction

We consider a cubic die whose faces are numbered from 1 to 6.
If we roll this die 3 times in a row, then each possibility is a triplet (a, b, c) where a, b and c are chosen from the elements of the set $E = \{1, 2, 3, 4, 5, 6\}$.
Any possibility is called arrangement with repetition of 3 elements out of 6 (possibly an element of the set E can appear several times).
According to the basic principle of counting, the total number of possibilities is:

$$N = 6 \times 6 \times 6 = 6^3$$

Property

Let E be a non-empty finite set of cardinal (number of elements) $n \geq 1$ and $p \in \mathbb{N}^*$ such that: $1 \leq p \leq n$.
The number of arrangements with repetition of p elements of E is the number denoted n^p.

Example

The number of 8 symbol passwords that can be created with 66 characters is: 66^8.
Each password represents a repetitive arrangement of eight items chosen from 66.

2. Arrangement without repetition

Introduction

We want to form a number n of 3 digits $n = c\,d\,u$ chosen from 1, 2, 3, 4, 5 and 6.

In addition, all figures are required to be separate two by two. Any possibility is a triplet (a, b, c) where a, b, and c are distinct two by two and chosen from my elements of the set
E = {1, 2, 3, 4, 5, 6}.
Any possibility is called arrangement without repetition of 3 elements out of 6.
According to the basic principle of counting, the total number of possibilities is:

$$N = 6 \times 5 \times 4 = 120.$$

This number is noted: A_6^3, therefore: $A_6^3 = 6 \times 5 \times 4 = 120.$

Property:

Let E be a non-empty finite set of cardinal (number of elements) $n \geq 1$ and $p \in \mathbb{N}^*$ such that: $1 \leq p \leq n$.
The number of arrangements without repetition of p elements of E is the number noted A_n^p given by:

$$A_n^p = n(n-1)(n-2)............(n-p+1).$$

Example

1. A club of 32 numbers must elect from among its numbers its bureau consisting of a president, a secretary and a treasurer.
   Question: What is the number of possible desks?
   Each desk can be thought of as a non-repeat arrangement of 3 objects taken from 32 objects, and therefore the number of possible desks is: N = $A_{32}^3 = 32 \times 31 \times 30 = 29760.$
2. How many football teams (eleven players) can be formed with the twenty-six members of the team, taking into account the place of the days?
   Each team can be thought of as a non-repeat arrangement of 11 items chosen from 26 items.
   Consequently, the number of teams requested is: A_{26}^{11}.

Note:
If n = p, then any arrangement without repetition of n elements among n is called a permutation to n elements.
The number of permutations with n elements is: $A_n^n = n \times (n-1) \times (n-2) \times .... \times 2 \times 1.$
This number is denoted by n! which we read as factorial n, therefore: $n! = 1 \times 2 \times 3 .... \times n$.

Example:

1. The number of possibilities to store 5 books in 5 drawers so that each drawer can hold no more than one book is: 5!
2. The number of ways to place 30 students on 30 chairs in a room is the number of ways to swap the 30 students on the 30 chairs, i.e. 30!

Note:

- ▶ We agree that: 0! = 1 and $A_n^0 = 1$.
- ▶ We have for all $(n; p) \in \mathbb{N}^2$ such that $1 \leq p \leq n$: $A_n^p = \dfrac{n!}{(n-p)!}$.

  In particular, we have: $A_n^1 = n$ and $A_n^n = n!$

Application

1. Determine the number of natural numbers of three distinct digits two by two, made up of the digits: 1, 2, 3, 4, 5, 6 and 7.
2. A bag contains 11 balls divided as follows: 4 black, 5 blue and 2 yellow.
   3 balls of bag are drawn successively and without delivery.
   a- How many draws are possible?
   b- What is the number of possibilities to obtain :
      i- Three black balls?
      ii- Three balls of the same color?
      iii – Two black balls and only one white ball?
      iv – A black, a blue and a yellow in that order?
3. With the letters of the word "IZENZARN" used once each, how many words can be formed (with or without meaning)?
4. Calculate: A_{10}^8, A_{20}^7, A_{2025}^1, A_n^2, A_{2025}^{2025}.

3 Combination

Introduction

Let p_1, p_2, p_3 and p_4 four people, we must form a team of 2 people to participate in a meeting. The possible choices are: $\{p_1;p_3\},\{p_1;p_2\},\{p_1;p_4\},\{p_2;p_3\},\{p_2;p_4\}$. So the total number of possibilities is 6, it is the combination of 2 elements out of 6, so denoted C_6^2, $C_6^2=6$.

Property

The number of combinations of p elements of p elements taken from n distinct elements is the number of sets of p objects taken from the n starting objects. This number is noted:

$C_n^p=\frac{A_n^p}{p!}=\frac{n!}{p!(n-p)!}$.

Example

1. When 4 jobs are recruited, 8 men and 6 women are recruited.
   Determine the number of possible distinct recruitments:
   There are 8 + 6 = 14 possible candidates. So the number of separate recruitments is:
   $C_{14}^4=\frac{14!}{4!(14-4)!}=1001.$
2. An urn contains 4 red balls, 3 black balls and a white ball. Three balls are drawn simultaneously from the urn.
   a- The number of possible draws is: $C_8^3=\frac{8!}{3!(8-3)!}=56.$
   b- The number of sorts with exactly 2 black balls is: $C_3^2\times C_3^1=15.$
   c- The number of draws with a ball of each color is: $C_4^1\times C_3^1\times C_1^1=12.$

Note:

▶ Let n and p be two natural numbers such that: p < n. We have the following properties:

- Symmetry: $C_n^p=C_n^{n-p}$. In particular: $C_n^0=C_n^n=1$ and $C_n^1=C_n^{n-1}=n$.
- Pascal's formula: $C_n^p+C_n^{p+1}=C_{n+1}^{p+1}$.

▶ Let a and b be two reals and n a nonzero natural number. We then have the following formula, called

"Newton's binomial formula": $(a+b)^n=\sum_{k=0}^{n}C_n^k a^k b^{n-k}$.

Application:

In a class of 32 students, there are 19 boys and 13 girls. Two delegates must be elected.

1. How many choices are possible?
2. What is the number of choices if we impose a boy and a girl?
3. What is the number of choices if we impose two boys?

Different types of prints

Types of Draw	Order	Number of possible draws
Simultaneous	The order does not intervene	C_n^p
Successive without postponement	The order of the	A_n^p
Successive with discount	The order of the	n^p

III Vocabulary of Probability

- **Random experiment:** An experiment whose result cannot be known with certainty.
- **Universe of possibilities:** The set of all the possibilities of a random experiment that we denote Ω. We set $\Omega=\{a_1;a_2;........;a_n\}$, n is called the cardinal of Ω, we write: n = card (Ω).
- **Event:** Is any part of A of Ω. Singletons of Ω (parts formed from a single element) are called elementary events.

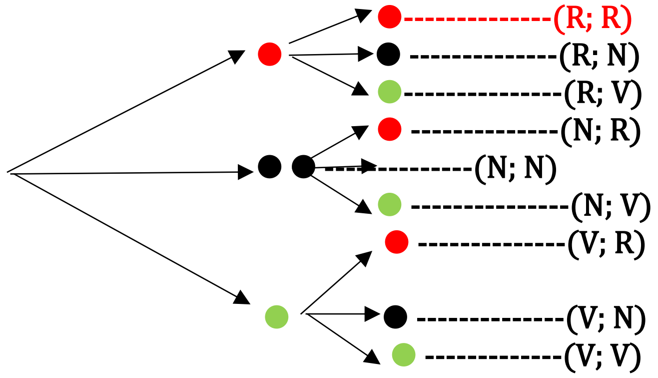
- Contrary event: Of an event A denoted $\overline{A}$ is composed of the elements of 'Ω that are not in A. An event A is said to be realized if the result of the experiment gives rise to a possibility belonging to A. ϕ is called an impossible event and 'Ω a certain event.
- Incompatible events: Two events A and B are said to be incompatible if and only if: $A \cap B = \phi$.

  We have: $A \cap \overline{A} = \phi$ and $A \cup \overline{A} = \Omega$.
- Event $A \cap B$: is event A and B.
- Event $A \cup B$: is event A or B.

Example:

An urn contains three balls: A red ball, a black ball and a green ball: written in series R, N and V. We randomly draw 2 balls successively and with put back in the urn.

1. Draw up a weighted tree suitable for this random experiment:
   A tree allows you to model a situation, you can also visualize all the possible outcomes of this random experiment using the tree below.



2. Determine the universe of possibilities 'Ω of this experiment:
   'Ω = {(R; R) ; (R; N) ; (R; V) ; (N ; R) ; (N ; N) ; (N ; V) ; (V; R) ; (V; N) ; (V; V)}
   We therefore have: Card ('Ω) = 9.
3. The following events are considered:
   - ◆ A: "drawing of two balls of different colors"
   - ◆ B: "the first ball drawn is red"
   - ◆ C: "the two balls drawn are red"
   - ◆ D: "draw at least one black ball"

   a- Determine in detail the events A, B, C and D:

   We have: A = {(R; N), (R; V), (N ; R), (N ; V), (V; R), (V; N)}. And we have card (A) = 6.
   We have: B = {(R; N), (R; V), (R; R)}. And we have: card (B) = 3.
   We have: C = {(R; It is an elementary event. And we have card (C) = 1.
   We have: D = {(N ; R), (R; N), (N ; V), (V; N), (N ; N)}. And we have card (D) = 5.

   b- Determine in detail the following events: $A \cap B$, $C \cap D$, $A \cup B$ and $\overline{D}$.

   - ■ We have: $A \cap B = \{(R;N),(R;V)\}$. And we have: $card\left(A \cap B\right) = 2$.
   - ■ We have: $A \cap C = \phi$. A and C are two incompatible events. And we have: $card\left(A \cap C\right) = 0$.
   - ■ We have: $A \cup B = \{(R;N),(R;V),\ (N;R),(N;V),(V;R),\ (V;N),(R;R)\}$. And we have:

   $Card\left(A \cup B\right) = 7$.
   - ■ We have: $\overline{D} = \{(R;R),\ (V;V),(R;V),(V;R)\}$. And we have: $card\left(\overline{D}\right) = 4$.

Application

A ballot box contains 3 red labels numbered with the numbers 1, 2, 3 and two black labels numbered with the numbers 4, 5.

Two labels are drawn at random successively and without handing over from the ballot box.

1. Determine in detail the universe of possibilities of 'Ω.
2. Determine in detail the following events:
   A: "printing of two labels of different colours"
   B: "printing of two labels whose sum of digits is equal to 4"

C: "the first label drawn is numbered with the number 2"
D: "printing of two labels whose sum of the digits is less than or equal to 4"
E: "print run at least one black label"

3. Determine in detail the opposite event $\overline{A}$ of event A.
4. Determine the event in detail $A \cap C$.
5. Determine the event in detail $A \cup C$.
6. Show that the two events A and B are incompatible.

Property
Let Ω be the universe linked to a random experiment and let A and B be two events of Ω. We have:

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$;
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$;
- A $= (A \cap B) \cup (A \cap \overline{B})$.

IV Concept of probability

1 Probability of an event

Definition
To study a statistical population, we determine the frequencies f, for all possible outcomes of character x_i.
If the number of elements of the population is large enough, the frequency of each possible result stabilizes at any number f_i which belongs to the interval [0; 1].
If we randomly choose an element x_i of population. The probability of getting x_i is the frequency f_i and
we write: $f_i = P_i$.
P_i is read probability of the elementary event $\{x_i\}$.

Results
Let Ω be a finite universe of possibilities of a random experiment.
We set Ω $= \{x_1, x_2, ..., x_n\}$.

- $f_1 + f_2 + ... + f_n = 1$, hence $P_1 + P_2 + ... + P_n = 1$.
- The probability of an event A is equal to the sum of the probabilities of the events elementary that it forms, it is denoted P (A).

<u>Example:</u>
A loaded cubic die is rolled with its faces numbered from 1 to 6. It is assumed that the probability of 6 occurring is twice the probability of each of the numbers 1, 2, 3, 4, and 5.

1. Determine the probability of each elementary event: We roll the die once, then the universe of possibilities is Ω = {1, 2, 3, 4, 5, 6} and for i ∈ Ω we set P_i is the probability of the elementary event: "appearance of a face bearing the number i".
   So we have: $P_6 = 2P_1 = 2P_2 = 2P_3 = 2P_4 = 2P_5$ and $P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 1$.
   We therefore conclude that: $P_1 = P_2 = P_3 = P_4 = P_5 = \frac{1}{7}$ and $P_6 = \frac{2}{7}$.
2. Determine the probability of each of the following events:
   a- A: "get an even number"
   We have: A = {2; 4; 6}, then P (A) = $P_2 + P_4 + P_6 = \frac{1}{7} + \frac{1}{7} + \frac{2}{7} = \frac{4}{7}$.
   b- B: "get an odd number less than or equal to 3"
   We have: B = {1; 3}, then P (B) = $P_1 + P_3 = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$.
   c- C: "obtain an even number strictly greater than 3"
   We have: C = {4; 6}, then P (C) = $P_4 + P_6 = \frac{3}{7}$.
   d- D: "get a multiple of 3 or an even number"
   We have: D = {2; 3; 4; 6}, then: P (D) = $P_2 + P_3 + P_4 + P_6 = \frac{5}{7}$.

Application

A balanced coin is thrown three times in a row.

1. List all possible outcomes.
2. Determine the probability of each of the following events:
   A: "The draw only has piles"
   B: "The draw has at least one heads".

2. Assumption of equiprobability

Definition

Any situation in the probabilities of all elementary events is equal.

Example

When we throw a well-balanced coin, we throw an unloaded die or we make a random draw of the balls indistinguishable to the touch, the outcomes have the same probability of realization, we say that we are in the presence of a situation of equiprobability.

Theorem

Under the assumption of equiprobability, the probability of an event A is: $P(A)=\dfrac{Card(A)}{Card(\Omega)}$ such that:

- ◆ Card (A) is the number of cases that occur event A.
- ◆ Card (Ω) is the number of possible cases.

Example:

An urn contains 4 red balls numbered 1 – 1 – 2 – 2 and three black balls numbered 1 – 2 – 2

It is assumed that all the balls are indistinguishable to touch.

3 balls from the urn are drawn at random and simultaneously.

1. Determine the number of possible draws: Since the draw of the 3 balls is carried out simultaneously, then each possibility is a combination of three elements out of 7 hence: $Card(\Omega)=C_7^3=35.$
2. Calculate the probability of each of the following events:
   a- A: "drawing of 3 balls of the same color":
   For event A to take place, you have to draw 3 red balls or 3 black balls.
   Hence: $P(A)=\dfrac{Card(A)}{Card(\Omega)}=\dfrac{C_4^3+C_3^3}{35}=\dfrac{4+1}{35}=\dfrac{5}{35}=\dfrac{1}{7}.$
   b- B: "draw of 3 balls that carry the same number":
   For event B to occur, 3 balls with the number 1 or 3 balls with the number 2 must be drawn.
   Hence: $P(B)=\dfrac{Card(B)}{Card(\Omega)}=\dfrac{C_4^3+C_3^3}{35}=\dfrac{1}{7}.$
   c- C: "drawing of a red ball only":
   For event C to occur, you must draw a single red ball and two black balls.
   Hence: $P(C)=\dfrac{Card(C)}{Card(\Omega)}=\dfrac{C_4^1\times C_3^2}{35}=\dfrac{12}{35}.$
   d- D: "draw of at least one ball with the number 2":
   For the event D to occur, one ball must be drawn with the number 2 and two balls with the number 1, or two balls with the number 2 and one ball with the number 1, or three balls with the number 2.
   Hence: $P(D)=\dfrac{Card(D)}{Card(\Omega)}=\dfrac{C_3^1\times C_4^2+C_3^2\times C_4^1+C_3^3}{35}=\dfrac{18+12+1}{35}=\dfrac{31}{35}.$
   e- E: "drawing of a red ball with the number 1":
   For the event E to occur, a single red ball must be drawn with the number 1.

Hence: $P(E)=\dfrac{Card(E)}{Card(\Omega)}=\dfrac{C_2^1\times C_5^2}{35}=\dfrac{20}{35}=\dfrac{4}{7}$.

Application

A case contains 3 red balls, 5 green balls and 2 black balls.
All the balls are indistinguishable to the touch.
3 balls are drawn at random and without remittance.

1. Determine the number of possible draws.
2. Determine the probability of each of the following events:
   a- A: "Draw of 3 balls of the same color"
   b- B: "Draw of 3 balls of different colors"
   c- C: "Sorting a red ball in the second draw"
   d- D: "Draw at least one green ball"
   e- E: "Draw of a maximum of one black ball".

Property

Let Ω be the universe linked to a random experiment and P is a probability defined on Ω. Let A and B be two events of Ω. We have:

- $P(\Omega)=1$ and $P(\phi)=0$. ◆ $0\leq P(A)\leq 1$.
- If A and B are incompatible, then $P(A\cup B)=P(A)+P(B)$.
- $P(\overline{A})=1-P(A)$ and $P(B\cap\overline{A})=P(B)-P(B\cap A)$.
- $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.

Example:

An urn contains 4 red balls numbered 1 – 2 – 2 – 2 and 3 black balls numbered 2 – 2 – 3.
It is assumed that all the balls are indistinguishable to touch.
A ball is randomly drawn from the urn.
The following events are considered:

A: "drawing of a black ball"; B: "drawing of a ball that bears the number 2"
C: "drawing of a red ball"; D: "drawing of a ball that bears the number 3"
E: "drawing of a ball that bears an odd number".

1. Calculate the following probabilities: P (A), P (B) and $P(A\cap B)$.

   We randomly draw a ball from the urn, then the universe of possibilities contains 7 possibilities, that is: Card (Ω) = 7.
   All the balls are indistinguishable to touch so we have the assumption of equiprobability hence:
   $P(A)=\dfrac{Card(A)}{Card(\Omega)}=\dfrac{3}{7}$; $P(B)=\dfrac{Card(B)}{Card(\Omega)}=\dfrac{5}{7}$ and $P(A\cap B)=\dfrac{Card(A\cap B)}{Card(\Omega)}=\dfrac{2}{7}$.

2. From this, deduce the probability of the event $(A\cup B)$:

   We have equality: $P(A\cup B)=P(A)+P(B)-P(A\cap B)$, then: $P(A\cup B)=\dfrac{3}{7}+\dfrac{5}{7}-\dfrac{2}{7}=\dfrac{6}{7}$.

3. Calculating Probability $P(D\cap C)$:

   We have: $C\cap D=\phi$ that is to say that the two events C and D are incompatible.

   And we have: $P(C)=\dfrac{Card(C)}{Card(\Omega)}=\dfrac{4}{7}$ and $P(D)=\dfrac{Card(D)}{Card(\Omega)}=\dfrac{1}{7}$. Hence

   $P(C\cap D)=P(C)+P(D)=\dfrac{5}{7}$.

4. Calculate the probability P (E):

   The opposite event of event E is B, hence: $P(E)=1-P(\overline{E})=1-\dfrac{5}{7}=\dfrac{2}{7}$.

Application

An apparatus, manufactured in very large series, may present two kinds of defects designated by D_1 and D_2.
In a batch of 1000 devices, it can be seen that 70 devices have the D_1 defect, 60 devices have the D_2 defect and 30 devices have both defects.
A customer buys a device (randomly).
Determine the probability of the following events:
◆ A: "The device has both defects"; ◆ B: "The device has at least one defect"
◆ C: "The device has no defects"; ◆ D: "The device has the defect D_1 and does not have the defect D_2"
◆ E: "The device has the defect D_2 and does not have the defect D_1"; F: "The device has only one defect".

V Conditional Probability: Independence

1 Conditional probability

Definition

Let A and B be two events of a universe of possibilities Ω such that P (B) ≠ 0.
The conditional probability of the event "A if B" or "A knowing B" is called the quotient $\frac{Card(A\cap B)}{Card(B)}$ denoted $P_B(A)$ or P (A/B) i.e.: $P\left(A/B\right)=\frac{P(A\cap B)}{P(B)}$.

Example:

A case contains 5 red balls, 4 black balls and 3 green balls. It is assumed that all the balls are indistinguishable to touch.
2 balls are drawn at random and without remittance.

1. Determine the number of possible outcomes:
   Each print is an arrangement without repetition of two elements out of 12, hence:
   $Card(\Omega)=A_{12}^2=132$.
2. Calculate the probability of the following events:
   a- A: "The first ball drawn is red":
   $$P(A)=\frac{Card(A)}{Card(\Omega)}=\frac{A_7^1\times A_5^1+A_5^2}{132}=\frac{5}{12}.$$
   b- B: "The second ball drawn is black, knowing that the first is red":
   We consider the following event D: "the second ball drawn is black"
   We have: $P(B)=P\left(D/A\right)=\frac{P(A\cap D)}{P(A)}=\frac{A_5^1\times A_4^1}{A_7^1\times A_5^1+A_5^2}=\frac{4}{11}.$
   c- C: "The second ball drawn is green, knowing that the first is red":
   We consider the following two events: E: "the second ball drawn is green" and
   F: "The first ball drawn is black"
   We have: $P(C)=P\left(E/F\right)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{A_3^1\times A_4^1}{132}}{\frac{A_4^1\times A_8^1+A_4^2}{132}}=\frac{3}{11}.$

Theorem

Let A and B be two events of non-zero probabilities, we have:
◆ $P(A\cap B)=P(B)\times P\left(A/B\right)$. ◆ $P(A\cap B)=P(A)\times P\left(B/A\right)$.

Application:

A health center proposes to screen a population of 1000 individuals for a disease.
The following data are available:
■ The proportion of people who are sick is 10%.
■ Out of 100 sick people, 98 have a positive test.

■ Out of every 100 people who are not sick, only one person has a positive test.
■ A person is randomly selected and tested.
■ We note M: "the person is sick" and T: "the person has a positive test"
■ E: "The device has only one defect."

1. Determine the probability that a sick person will test positive, as well as the probability that a sick person will have a negative test.
2. Determine the probability that a non-sick person will have a negative test, as well as the probability that a non-sick person will have a positive test.
3. Determine the probability of the following events:
   - E: "the chosen person has a positive test"
   - F: "the chosen person is sick knowing that he or she has a negative test"

2 Independence of two events

Definition

Let A and B be two events of non-zero probabilities.
A and B are said to be independent when the realization of one does not affect the other.

Theorem

Two events A and B of non-zero probabilities are independent if and only if they satisfy the following condition: $P(A\cap B)=P(A)\times P(B)$.

Note:

▶ A and B are two independent events, so $P\left(A/B\right)=P(A)$ and $P\left(B/A\right)=P(B)$.

▶ Do not confuse independent events with incompatible events.

Example

An unloaded die is rolled once on a horizontal table, its faces are numbered from 1 to 6. The following two events are considered:
◆ A: "Obtain a multiple of 2"; ◆ B: "Obtain a multiple of 3".
Are the two events A and B independent?
We have: A = {2 , 4, 6} ; B = {3 , 6} and $A\cap B=\{6\}$.

Then: $P(A)=\frac{Card(A)}{Card(\Omega)}=\frac{1}{2}$; $P(B)=\frac{Card(B)}{Card(\Omega)}=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{6}$.

Then: $P(A\cap B)=\frac{1}{6}=P(A)\times P(B)=\frac{1}{2}\times\frac{1}{3}$, Hence A and B are independent.

Application

A case contains 6 balls numbered 1, 1, 1, 2, 3, 4. It is assumed that all the balls are indistinguishable to touch. A ball is drawn from the cash register at random.

1. Calculate the probabilities of each of the following two events:
   A: "to obtain a ball that has an even digit"
   B: "to obtain a ball with a number less than or equal to 3"
2. Are A and B independent?

3 Independence of the tests

Definition

We consider a random experiment consisting of n successive tests. If the outcomes of one test do not influence the results of subsequent tests, the random experiment is said to be independent tests.

Example:

The following experiments consist of independent tests:

▶ Throw a coin n successive times $n \geq 2$ (each throw of the coin is a test).
▶ Throw of a successive two times $n \geq 2$.

Property

Let A be an event of probability p in a random test, and n a nonzero natural number.

When we repeat this test n times identically and independently, then the probability that event A is realized exactly k times is: $C_n^k p^k (1-p)^{n-k}$ where k ∈ {0; 1; 2; 3; ....; n}.

Example:

We roll a well-balanced cubic die 10 times in a row with faces numbered from 1 to 6 and we are interested in the number of 6 obtained.

- The probability of getting "number 6" exactly three times is: $C_{10}^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$.
- The probability of ever getting "the number 6" is: $C_{10}^0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$.
- The probability of getting "the number 6" exactly once is: $C_{10}^1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9$.

VI The random variable; probability distribution

1. The random variable

Definition

Let be a random experiment with a finite universe of issues.
If we associate each outcome of the random experiment with a unique real number, we say that we have defined a relation from 'Ω to R called a random variable generally denoted X.

Rating:

- The set X ('Ω) denotes the set of values taken by X.
- The event "X takes the value x_i" is denoted $(X = x_i)$.
- The event "X takes values less than or equal to x_i" is denoted $(X \leq x_i)$.
- The event "X takes values greater than or equal to x_i" is denoted $(X \geq x_i)$.
- The event "X takes the values between x_i and x_j" is denoted $(x_i \leq X \leq x_j)$.
- The probability of the event $(X = x_i)$ is denoted $P(X = x_i)$.

Example:

You flip a well-balanced coin three times in a row and note the result.

1. Determine the total number of outcomes:
   $\Omega = \{(PPP),(PPF),(PFP),(PFF),(FPP),(FPF),(FFP),(FFF)\}$.
2. Let X be the random variable that at each elementary event associates the number side "heads" obtained.
   a- Determine the possible values of X:
   In this random experiment the face can never be obtained, 1 time, 2 times or 3 times, hence the possible values of X are: 0, 1, 2 and 3 and we write: X ('Ω) = {0,1,2,3}.
   b- What is the probability of the event (X = 0)?
   The probability of the event (X = 0) is: $P(X = 0) = \frac{1}{8}$.
   c- What is the probability of the event (X = 1)?
   The probability of the event (X = 1) is: $P(X = 1) = \frac{3}{8}$.
   d- What is the probability of the event (X = 2)?
   The probability of the event (X = 2) is: $P(X = 2) = \frac{3}{8}$.
   e- What is the probability of the event (X = 3)?
   The probability of the event (X = 3) is: $P(X = 3) = \frac{1}{8}$.

2 Probability distribution of a random variable

Definition

'Ω is a universe of possibilities of a random experiment.
X is a random variable defined on 'Ω that takes the values $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$.

Defining the probability distribution of X means associating the probability of the event ($X = x_i$) with each value x_i (with $1 \le i \le n$).

Note

If we want to determine the probability distribution of a random variable X, we first determine the universe of outcomes Ω, count all the possible values of X, calculate all the probabilities of events ($X = x_i$) and present the results in a table in the following form:

x_i	x_1	x_2	x_3	…………	x_{n-2}	x_{n-1}	x_n
$P(X = x_i)$	P_1	P2	P_3	…………	P_{n-2}	P_{n-1}	P_n

Example:

An urn contains 10 balls numbered from 0 to 9. It is assumed that all the balls are indistinguishable to touch.

The crate is randomly drawn and simultaneously 3 balls.

X is the random variable that gives the number of balls drawn that carry an even number.

1. Determine the number of possible outcomes:
   The number of possibilities is: Card (Ω) = $C_{10}^{3} = 120$.
2. Write in detail the set X (Ω):
   The possible values of X are:
   ▣ 0 if the three balls drawn have an odd number.
   ▣ 1 if among the 3 balls drawn only one that bears an even number.
   ▣ 2 if among the 3 balls drawn only one that has an odd number.
   ▣ 3 if the three balls drawn have an even number.
   We therefore write: X (Ω) = {0; 1; 2; 3}.
3. Determine the probability distribution of the random variable X: We have:
   - $P(X = 0) = \frac{Card(X=0)}{Card(\Omega)} = \frac{C_5^3}{120} = \frac{10}{120} = \frac{1}{12}$.
   - $P(X = 1) = \frac{Card(X=1)}{Card(\Omega)} = \frac{C_5^1 \times C_5^2}{120} = \frac{50}{120} = \frac{5}{12}$.
   - $P(X = 2) = \frac{Card(X=2)}{Card(\Omega)} = \frac{C_5^2 \times C_5^1}{120} = \frac{50}{120} = \frac{5}{12}$.
   - $P(X = 3) = \frac{Card(X=3)}{Card(\Omega)} = \frac{C_5^3}{120} = \frac{10}{120} = \frac{1}{12}$.

   The following table gives the probability distribution of the random variable X:

x_i	0	1	2	3
$P(X = x_i)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

4. What is the probability of the event ($X \le 2$)?
   The event ($X \le 2$) is: draw of 3 balls with an odd number or 2 balls with an odd number and a ball with an even number or a ball with an odd number and 2 balls with an even number.
   Hence: $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{12} + \frac{5}{12} + \frac{5}{12} = \frac{11}{12}$.

Application

An urn contains 3 red balls and 2 black balls. Two balls are drawn at random and successively and with handover. It is assumed that all the balls are indistinguishable to touch.

X is the random variable that gives the number of red balls drawn.

1. Determine the number of possible outcomes.
2. Determine the possible values of X.
3. Determine the probability distribution of the random variable X.
4. What is the probability of the event ($X \le 2$)?

VII Mathematical Expectation; Variance; standard deviation

1 Mathematical expectation

Definition

The mathematical expectation of a random variable is intuitively the value that we expect to find, on average, if we repeat the same random experiment a large number of times.
It is denoted E (X), reads "Expectation of X" and is given by the following relation:
$E(X) = x_1p_1 + x_2p_2 + x_3p_3 + ... + x_np_n$.

Example:

If we consider the probability distribution of the following random variable:

x_i	0	1	2	3
$P(X = x_i)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

The mathematical expectation of X is: $E(X) = 0\times\frac{1}{12} + 1\times\frac{5}{12} + 2\times\frac{5}{12} + 3\times\frac{1}{12} = \frac{3}{2}$.

2 Variance

Definition

The variance of a random variable X is denoted V (X) and is given by the following relationship:

$$V(X) = \left(p_1{x_1}^2 + p_2{x_2}^2 + ... + p_n{x_n}^2\right) - \left[E(X)\right]^2.$$

Example:

In the probability distribution of the random variable X above:

The variance of X is: $V(X) = 0^2\times\frac{1}{12} + 1^2\times\frac{5}{12} + 2^2\times\frac{5}{12} + 3^2\times\frac{1}{12} - \left(\frac{3}{2}\right)^2 = \frac{7}{12}$.

3 Standard deviation

Definition

The standard deviation of a random variable X, denoted σ, is the square root of the variance. We have:

$$\sigma = \sqrt{V(x)}.$$

Note:
The standard deviation gives the dispersion of the values of X with respect to the expectation.

Application

An urn contains 7 white balls and 3 black balls.
We draw a ball at random and repeat the operation with a discount.
Let X be the number of white balls obtained in 6 draws.

1. Specify the distribution X, we will give the values taken by X and for each of these values of P (X = k).
2. Calculate the probability of getting:
   a- 5 white balls in 6 draws.
   b- At most 4 white balls in 6 draws.
   c- No more than 5 balls and no less than 2 balls in 6 draws.
3. Determine the mathematical expectation E (X) and the variance V (X).

EXERCISES AND PROBLEMS

Exercise 1

Let A and B be two events associated with the same random experiment. Questions 1, 2 and 3 are independent.

1. In this question, we assume that: P (A) = 0.7 and P (B) = 0.85 and $P(A\cup B)=0,9$.

   a- Calculate: $P(A\cap B)$, $P(\overline{A})$, $P(\overline{B})$ and $P(\overline{A\cup B})$.

   b- Calculate: $P(A\cap\overline{B})$, $P(\overline{A}\cap B)$, $P(A/B)$ and $P(B/A)$.

   c- Are events A and B independent? justify.

2. In this question, it is assumed that: $P(\overline{A})=0,36$; $P(\overline{B})=0,74$ and $P(\overline{A\cup B})=0,27$.

   a- Calculate: $P(A\cap B)$, $P(A/B)$ and $P(B/A)$.

   b- Calculate: $P(\overline{A}\cap B)$, $P(A\cap\overline{B})$, $P(\overline{A}\cap\overline{B})$ and $P(\overline{A}\cup\overline{B})$.

   c- Calculate: $P(B/\overline{A})$, $P(A/\overline{B})$ and $P(\overline{A}/\overline{B})$.

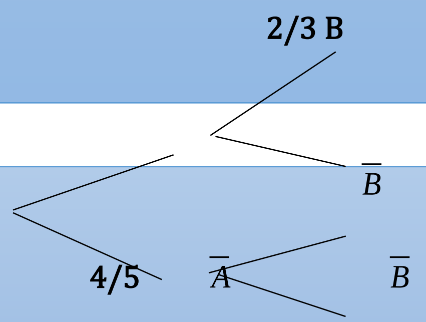
   d- Are events A and B independent? justify.

3. In this question, it is assumed that: $P(A\cap B)=0,3$, $P(\overline{B})=0,58$ and $P(B/A)=\frac{6}{13}$.

   a- Calculate: $P(\overline{A}\cap B)$.

   b- Calculate: P (A) and $P(A/B)$.

   c- Calculate: $P(A\cup B)$.



Exercise 2 A

Let A and B be two events associated with a random experiment.
Consider the following tree: B

1. Calculate P (A) and $P(\overline{B}/A)$.

2. Complete the tree with the missing probabilities knowing that: $P(\overline{A}\cap\overline{B})=\frac{1}{20}$.

3. Calculate P (B).
4. Are events A and B independent?

Exercise 3

A disease affects 1% of a given population. A drug test gives the following results:

- In sick individuals, 99% of tests are positive.
- In non-sick individuals, 98% of tests are negative.

An individual from this population is randomly selected. The following events are considered:

- T: "The test is positive"
- M: "The individual chosen is sick"

1. Build a weighted tree, and calculate $P(\overline{M})$, $P(\overline{T})$ and $P(T/\overline{M})$.

2. Calculate $P(T\cap M)$ $P(\overline{M}\cap T)$ and deduce P (T).

3. Knowing that the test is positive, what is the probability that this individual is sick?

Exercise 4

An S1 bag contains 5 red balls and 3 black balls. An S_2 bag contains 2 red balls and a black ball. All the balls are indistinguishable to the touch.

One of the bags is randomly selected and then a ball is randomly drawn from it. (There is an equiprobability in the choice of bags). We consider the events:

- ■ S_1: "Choose the bag S_i where $i \in \{1 ; 2\}$ »
- ■ R: "Get a red ball"
- ■ N: "Get a black ball".

The S_1 bag The S_2 bag

1. Build a weighted tree.
2. Calculate the probability of choosing the bag S_2 and obtain a red ball.
3. Calculate the probability of getting a red ball.
4. Knowing that the ball drawn is black, what is the probability that it is drawn from the bag S_1?

Exercise 5

▣ An S1 bag contains 2 tokens with the number 1 and 4 tokens with the number 2.
▣ An S_2 bag contains 3 red balls and 4 green balls.
All balls and tokens are indistinguishable by touch.

The S_1 bag The S_2 bag

1. A token of S 1 is drawn at random.
   Calculate the probability of each of the following events:
   ▶ A: "The drawn token is numbered 1"
   ▶ B: "The token drawn is numbered 2"
2. The following random experiments are considered:
   We draw a token of S_1 at random and write down its number:
   ◆ If this number is 1, a ball of S 2 is randomly drawn.
   ◆ If this number is 2, two balls of S 2 are drawn at random and simultaneously.
   Let E_n be the event: "get exactly n red balls" where $n \in \{1; 2\}$.
   a- Show that: $P(E_1) = \dfrac{11}{21}$ and $P(E_2) = \dfrac{2}{21}$.
   b- Knowing that we get a single red ball, calculate the probability that the chip drawn from S_1 has the number 1.

Exercise 6

We consider a balanced cubic die whose faces are numbered from 1 to 6.
One bag contains 4 red balls and 3 green balls. All the balls are indistinguishable to the touch.

The bag The dice

Two balls from the bag are randomly and simultaneously drawn and then the die is rolled.
Calculate the probability of each of the following events:

▲ A: "get two red balls and the n ⁰ 5"
▲ B: "get two green balls and an even number"
▲ C: "get two balls of different colors and a multiple of 3"

Exercise 7

▶ A U 1 urn contains 5 red balls and three black balls.
▶ A U2 urn contains 4 red balls, 2 green balls and a black ball.
All the balls are indistinguishable to the touch.
Two balls from urn U_1 and one ball from urn U_2 are drawn at random and simultaneously.

The U_1 Urn The U 2 Urn

Calculate the probability of each of the following events:

▣ A: "get 3 balls of the same color"

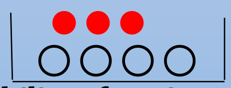
▣ B: "get 2 red balls exactly"

▣ C: "to obtain a ball of each color",

Exercise 8

An urn contains 3 red balls and 4 white balls. The balls are indistinguishable to the touch.
A ball is randomly drawn from the urn, its colour is noted, then it is put back in the urn: this test is repeated 9 times in a row. (Successive draw with 9 balls).

The urn

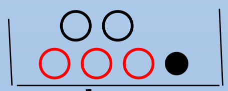


1. Calculate the probability of getting a red ball exactly twice.
2. Calculate the probability of getting a red ball exactly 6 times.
3. Calculate the probability of getting a red ball at least once.

Exercise 9

An urn contains 3 red balls, 2 white balls and a black ball.
All the balls are indistinguishable to the touch.

The urn



1. 3 balls are drawn at random successively and with handover.
   Calculate the probability of each of the following events:
   - A: "Obtain 3 balls of distinct color two by two (one ball of each color).
   - B: "get 3 balls of the same color"
   - C: "get 2 colors exactly"
   - D: "get 2 red balls exactly"
2. We now draw at random successively and without delivery 3 balls from the urn.
   Calculate the probability of each of the events A, B, C, and D.
3. We now randomly draw 3 balls from the urn simultaneously.
   a- Calculate the probability of each of the events A, B, C, and D.
   b- Let X be the random variable that at each draw associates the number of red balls remaining in the urn after the draw of 3 balls. Determine the probability distribution of X.

Exercise 10

A bag contains:

■ 3 white tokens numbered: 1; 1 ; 2.

■ 4 red tokens numbered: 1; 1 ; 2 ; 2.

■ 5 green tokens numbered: 1; 2 ; 2 ; 2 ; 3.

All tokens are indistinguishable by touch.

1. We draw at random successively and with delivery 4 tokens from the bag.
   a- Calculate the probability of getting 4 tokens of the same color.
   b- Calculate the probability of getting a single white token.
   c- Calculate the probability of getting 4 tokens with the same number.
   d- Calculate the probability of getting a single green chip and a single even number.
2. 4 tokens are drawn at random successively and without handing over from the bag.
   a- Calculate the probability of each of the following events:
      A: "get 4 tokens of the same color"
      B: "get a single red token"
      C: "get exactly two green chips numbered 2"
      D: "get 4 tokens with the same number"
      E: "get one red chip and one odd number"
   b- Let X be the random variable that at each draw of 4 chips associates the number of green chips that carry an odd number.
      Determine the probability distribution of X.

3. We now randomly draw 4 tokens from the bag simultaneously.
   Calculate the probability of each of the events A, B, C, D and E.
4. Consider the following game:
   Two tokens are drawn at random successively and without delivery. You win if you get two green chips with the number 2.
   Omar has played this game 3 times in a row (She puts the 2 drawn tokens back in the bag before the next draw).
   a- What is the probability that Omar will win 3 times?
   b- What is the probability that Omar will win exactly once?
   c- Omar wishes to play this game n times in a row so that the probability of the event G: "Omar wins at least once" is strictly greater than 0.999.
   Determine the smallest value of n.

Exercise 11

One urn contains three white balls and seven black balls. It is assumed that the balls have the same probability of being drawn.

1. Two balls from the urn are drawn at random and simultaneously. Let A and B be the two events:
   ▲ A: "the two balls drawn are black"
   ▲ B: "among the two balls drawn, there is at most one black ball"
   Show that: $P(A)=\frac{7}{12}$ and $P(B)=\frac{8}{15}$.
2. In this question, we consider the following random experiment:
   A ball is drawn from the urn; if this ball is white, we stop, and if this ball is black, we draw from the urn a second and last ball.
   We consider the two events:
   ▶ C: "get a white ball on the first draw"
   ▶ D: "to obtain a white ball"
   a- Calculate the probability of event C.
   b- Show that the probability of event D is $\frac{8}{15}$.

Exercise 12

An urn U_1 contains six balls bearing the numbers 0; 0; 1; 1; 1; 1; 2 and an urn U_2 contains five balls bearing the numbers 1; 1; 1; 2; 2.
It is assumed that the balls of the two urns are indistinguishable to touch.
The following experiment is considered:
We draw a ball from urn U_1 and we note the number at which door, then we put it in urn U_2, then we draw a ball from urn U_2 and we note the number b it carries.
The following events are considered:

- A: "the ball drawn from the urn U_1 bears the number 1"
- B: "the product ab is equal to 2".

1. a- Calculate P (A); the probability of event A.
   b – Show that $P(B) = \frac{1}{4}$ (we can use the tree of possibilities)
2. Calculate P (A/B); probability of event A knowing that event B is realized.
3. Let X be the random variable that associates to each result of the experiment, the product ab.
   a- Show that $P(X = 0) = \frac{1}{3}$.
   b- Give the probability distribution of X (Note that the values taken by X are 0, 1, 2 and 4).
   c- We consider the events:
   M: "the product ab is non-zero" and N: "the product ab is equal to 1".
   Show that the events M and N are equiprobable.