ECONOMICS

CHAPTER 7 NUMERICAL SEQUENCES

1. General

1.1 Definition:

A real sequence is any family of elements of R indexed to N or, equivalently, any map of N in R.

1.2 Vocabulary:

A sequence (U_n) is constant if there exists C $\in IR$ such that for all n $\in IN$, $u_n = C$. A sequence is stationary if it is constant from a certain rank.

We say that a sequence (U_n) is bounded from above (resp. bounded from below) if there exists C $\in IR$ such that for any natural number n, $u_n \leq C$ $(resp. u_n \geq C)$.

A sequence is said to be bounded if it is bounded from above and bounded from below.

1.3 Sense of variation:

Let (U_n) be a real sequence. Then

- (U_n) is increasing if and only if for any natural number n: $U_{n+1} - U_n \geq 0$.
- (U_n) is decreasing if and only if for any natural number n: $U_{n+1} - U_n \leq 0$.

Example 1:

Let (U_n) be a numerical sequence defined on N by: $U_n = 3n - 5$.

The sequence (U_n) is increasing because:

$U_{n+1} - U_n = 3(n+1) - 5 - 3n + 5 = 3n + 3 - 5 - 3n + 5 = 3 > 0$

Example 2:

Let (V_n) be a numerical sequence defined on N by: $U_0 = 2$ and $U_{n+1} = U_n - 5$.

The sequence (V_n) is decreasing because: $U_{n+1} - U_n = U_n - 5 - U_n = -5 < 0$.

Remark:

We can also define a strictly increasing, strictly decreasing or constant sequence.

Property 1: To study the variations of the sequence (U_n), it is sufficient to study the sign of $u_{n+1} - u_n$:

- If for all n, $u_{n+1} - u_n \geq 0$ then (U_n) is increasing.
- If for all n, $u_{n+1} - u_n \leq 0$ then (U_n) is decreasing.

Property 2: Let (U_n) be a sequence defined by a = f (n).

- If the function f is increasing over $[0;+\infty[$, then the sequence (U_n) is increasing.
- If the function f is decreasing on $[0;+\infty[$, then the sequence (U_n) is decreasing.

Note: The reciprocal of this property is false.

2. Arithmetic sequences

2.1 Definition, examples

Definition:

We say that a sequence (U_n) is arithmetic if we go from one term to the next by always adding the same real number r. We therefore have: The real r is then called the reason of the sequence.

Examples:

1. The sequence: 1, 6, 11, 16, 21, ... is arithmetic of reason 5.
2. The sequence defined by: $U_0 = 10$ and $U_{n+1} = U_n - 3$ is arithmetic of reason (-3).
3. The sequence of natural numbers: 0, 1, 2, 3, 4, 5, ... is arithmetic of reason 1.
4. The sequence of odd natural numbers is arithmetic of reason 2.

Property:

A sequence (U_n) is arithmetic if and only if the difference is constant for any integer n. In this case, the constant found is the reason for the sequence.

Theorem: Let (U_n) be an arithmetic sequence of reason r.

- If r > 0, then the sequence (U_n) is increasing.
- If r < 0, then the sequence (U_n) is decreasing.

Theorem:

Let (U_n) be an arithmetic sequence of reason r. Then: $u_n = u_0 + nr$.

$$u_n = u_1 + (n-1)r, \quad u_n = u_p + (n-p)r.$$

Sum of consecutive terms:

Theorem:

Let (U_n) be an arithmetic sequence of reason r. Let S_n be the sum of the first (n + 1) terms of the sequence (U_n), i.e.:

$$S_n = u_0 + u_1 + u_2 + \dots\dots + u_n = \left(\frac{n+1}{2}\right)(u_0 + u_n).$$

3. Geometric sequences

3.1 Definition, examples

Definition: A sequence (U_n) is said to be geometric if we move from one term to the next, always multiplying by the same real number q. We therefore have: $u_{n+1} = q \times u_n$.

The real q is then called the reason of the sequence.

Examples:

1. The sequence: 1, 2, 4, 8, 16, ... is geometric sequence of reason 2.
2. The sequence defined by: $u_0 = 3,\ u_{n+1} = -\frac{1}{2}u_n$ is geometric of reason $\left(-\frac{1}{2}\right)$.
3. The sequence defined by $u_n = (-1)^n$ is geometric of reason (-1).

Property: A sequence (U_n) is geometric if and only if the quotient $\frac{u_{n+1}}{u_n}$ is constant for any integer n. In this case, the constant found is the reason for the sequence.

Theorem: Let (U_n) be a geometric sequence of reason q. Then: $u_n = u_0 \times q$. $u_n = u_p \times (q)^{n-p}$.

Sum of consecutive terms

Theorem: Let (U_n) be a geometric sequence of reason q (where $q \neq 1$). We denote S_n the sum of the first (n +1) terms of the sequence (U_n), i.e.:

$$S_n = u_0 + u_1 + u_2 + \dots\dots + u_n = u_0\left(\frac{1-q^{n+1}}{1-q}\right).$$

Converging sequences:

A sequence (U_n) is said to admit a real L as a limit when any open interval containing L contains all the terms of the sequence from a certain rank.

We then denote $\lim\limits_{n \to +\infty} u_n = L$.

When a sequence (U_n) admits a finite limit, it is said to be convergent (or converge). Otherwise, it is said to be divergent.

Uniqueness of the boundary: If a sequence converges then its limit is unique.

Compatibility with order:

Let (U_n) and (V_n) be two convergent sequences of respective limits L and L'.

If, from a certain rank, we have a < v_n (or else $u_n \leq v_n$) $L \leq L'$.

Theorem

We consider three sequences (U_n), (V_n) and (W_n).

If (U_n) and (W_n) are convergent towards the same real L and if, from a certain rank, $u_n \leq v_n \leq w_n$ then (V_n) is also convergent towards L.

CHAPTER 6 NUMERICAL SEQUENCES EXERCISES

Exercise 1

Part I:

U1, One, r and S_n designating respectively the first term, the nth term, the reason and the sum of the first n terms of an arithmetic sequence, calculate:

1. U_n and S_n, knowing: $U_1 = 7$; $n = 120$; $r = 5$.
2. U_1 and S_n, knowing: $U_n = 326$; $n = 72$; $r = 2$.
3. r and S_n, knowing: $U_1 = 397$; $U_n = 64$; $n = 1000$.
4. n and U_n, knowing: $U_1 = 50$; $r = -4$; $S_n = 330$.
5. n and r, knowing: $U_1 = -3$; $U_n = 6$; $S_n = 28.5$.
6. U_1 and U_n, knowing: $n = 54$; $r = 4$; $S_n = 270$.

Part II

Calculate in the following case of a geometric sequence:

1. U_n and S_n, knowing: $U_1 = 2$; $q = 3$; $n = 5$.
2. q and S_n, knowing: $U_1 = 162$; $U_n = 32$; $n = 5$.
3. U_1 and S_n, knowing: $U_n = 54$; $q = 3$; $n = 4$.
4. U_1 and U_n, knowing: $q = 0.5$; $n = 7$; $S_n = 571.5$.

Exercise 2

1. Study the sense of variation of each of the following sequences defined by:
   a- $U_n = 3n - 8$; b- $V_n = -5n + 4$; c- $W_n = 7n - 3$.
2. What is the nature of the sequence (U_n)? Specify its first term U_0 and its reason.
3. Let the sequence (t_n) be defined by $t_0 = 2$ and $t_{n+1} = \frac{t_n + 4}{3}$.

   Let $K_n = t_n - 2$; show that (K_n) is a geometric sequence whose reason and first term will be determined.

Exercise 3

We consider the sequence (U_n) defined on IN by: $U_0 = -4$ and $U_{n+1} = \frac{3}{4}U_n + 1$.

1. Determine the first five terms of the sequence (U_n).
2. We set $V_n = U_n - \alpha$. $(\alpha \in IR)$.
   a) Determine α so that (V_n) is a geometric sequence.
   b) Deduce that for any natural number n; $U_n = 4 - 8\left(\frac{3}{4}\right)^n$.
   c) We denote $S_n = u_0 + u_1 + u_2 + \dots\dots\dots + u_n$. Find the expression of S_n in terms of n.
   d) Determine the limits of the sequences (U_n) and (S_n).

Exercise 4

We consider the sequence (U_n) defined on IN by: $u_0 = 2$ *and* $u_{n+1} = \frac{2}{3}u_n + 1$.

1. Calculate u_1, u_2, u_3.
2. Is the sequence (u_n) arithmetic? geometric?
3. We define the sequence (v_n) by: $v_n = u_n - 3$ for any natural number n.
   a) Calculate v_0, v_1, v_2.
   b) Determine the nature of the sequence (v_n).
   c) Deduce the expression of v_n as a function of n.
4. Express u_n in terms of v_n, then in terms of n. Calculate u_8.