LIFE SCIENCES CLASS

CHAPTER 9 PROBABILITIES

I- Random experiment – event

A random experiment is any experiment which, repeated under the same conditions, does not give the same results for each trial. The possible outcomes of this random experiment are called outcomes.
The set of outcomes is called the universe of random experience.
In the whole sequel, we will always place ourselves in the case where Ω is finished.
Any part of Ω is called an event.
The event Ø is called the impossible event and Ω is called the certain event.
An event that includes a single element is called an elementary event.
If A and B are two events.

- The event "A or B" is $A \cup B$, $A \cup B$ therefore corresponds to "A is realized or B is realized"
- The event "A and B" is $A \cap B$, $A \cap B$ therefore corresponds to "A is realized and B is realized"
- The opposite event of A is the complement of A in Ω, denoted $\overline{A}$.
- A and B are said to be incompatible if $A \cap B = \phi$.

A complete system of events of Ω is any finite family of events A_1, ..., A_n verifying:

- The events are two by two incompatible: ∀ i, j ∈ $\{1,......, n\}^2$, i ≠ j, $A_i \cap A_j = \phi$;
- Their meeting is Ω: $A_1 \cup A_2 \cup A_3 .... A_n = \Omega$.

II- Probability of two events coming together

Let A and B be two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

III - Contrary event:

Let be an event A. The probability of its opposite event is equal to: $P(\overline{A}) = 1 - P(A)$.

- $A \cup \overline{A} = \Omega$; $\quad A \cap \overline{A} = \phi$.

IV - Probability of an event

In a situation of equiprobability, the probability of an event A is equal to:
$P(A) = \dfrac{Card(A)}{Card(\Omega)}$ where card (A) = Number of elements of A and card (Ω) = Number of elements of Ω.

V - Conditional probabilities:

Definition:

We consider two events A, such that P (A) ≠ 0, and B.

We call the conditional probability of B knowing A the number: $P\left(B/A\right) = \dfrac{P(A \cap B)}{P(A)}$.

$$\Rightarrow P(A \cap B) = P\left(B/A\right) \times P(A) = P\left(A/B\right) \times P(B).$$

VI- Total probability formulas:

Definition:

We consider a non-zero natural number n.
The events A_1, A_2, ... , A_n form a complete finite system of events or a partition of the universe Ω if:

1. For everything $i \in \{1, 2, 3, ..., n\}$, $P(A_i) \neq 0$;
2. The events A_i are disjointed two by two; (i.e., $A_i \cap A_j = \phi$)
3. $A_1 \cup A_2 \cup A_3 \cup .... \cup A_n = \Omega$.

VII – Total probabilities – general case:

We consider the events $A_1, A_2, \ldots\ldots\ldots, A_n$ forming a complete system of finite events and an event B:
$P(B)=P(B\cap A_1)+P(B\cap A_2)+\ldots\ldots\ldots+P(B\cap A_n)$.

VIII – Independence:

Definition: Two events A and B are said to be independent if the probability of one does not change, regardless of whether the other occurs or not, i.e.: $P\left(B/A\right)=P(B)$ or $P\left(A/B\right)=P(A)$.

If A and B are two independent events, then: $p(A\cap B)=p(A)\times p(B)$.

IX – Combinations Let E be a set of n elements and p a natural number less than or equal to n.
Any part of E formed by p elements is called a combination of p elements distinct from E.
Such a combination is also called a combination of p elements among n, or a combination of n objects p to p. The number of these combinations will be noted C_n^P.

Example

$C_n^0=1$, Because E has a single part formed of zero elements (empty part).

$C_n^1=n$, For E has n parts formed of an element.

$C_n^n=1$, For E has a single part formed of n elements.

Let n and p be two nonzero natural numbers $(p\le n)$.

The number of combinations of n objects p to p is given by the formula:

$$C_n^p=\frac{n!}{p!(n-p)!}=\frac{n(n-1)(n-2)\ldots\ldots\ldots(n-p+1)}{p(p-1)(p-2)\ldots\ldots\ldots\ldots 2\times 1}.$$

For two natural numbers n and p $(p\le n)$, we have $C_n^P=C_{n-1}^p+C_{n-1}^{p-1}$.

X- Binomial formula: If a and b are two numbers and n is a nonzero natural number, then

$$(a+b)^n=a^n+C_n^1a^{n-1}b+\ldots\ldots\ldots+C_n^{n-1}ab^{n-1}+b^n.$$

XI - Discrete random variable and probability distribution

- Let us say Ω the universe associated with a random experiment. This is called a random variable defined on Ω any map of Ω in IR.
  When Ω is finite, the random variable is said to be discrete.
- The probability distribution of a random variable is any map p defined on $X(\Omega)=\{x_1,x_2,\ldots\ldots,x_n\}$ to values in [0; 1] such that:
  $p(X=x_1)+p(X=x_2)+\ldots\ldots+p(X=x_n)=1$.

XII- Distribution function

The distribution function of a random variable X is the function

$$F: IR\to[0,1]$$
$$x\to F(x)=p(X\le x).$$

XIII - Mean, Variance and Standard Deviation of a Random Variable

- Let $x_1, x_2, \ldots\ldots\ldots, x_n$ be the possible values of a random variable X and $p, \ldots, p_n$ their corresponding probabilities.
  The mathematical expectation of X is the real number denoted E(X) and defined by
  $$E(X)=p_1x_1+p_2x_2+\ldots\ldots\ldots\ldots+p_nx_n=\sum_{i=1}^{n}p_ix_i=\overline{X}.$$
- The variance of X, of mathematical expectation m, is the positive real number denoted Var(X) and defined by:
  $$Var(X)=p_1x_1^2+\ldots\ldots\ldots+p_nx_n^2-m^2=\sum_{i=1}^{n}x_i^2p_i-m^2.$$

- The standard deviation of a random variable X is the positive square root of its variance. We note it $\sigma_x = \sqrt{\operatorname{var}(x)}$.

Example 1

In a company, we are interested in the distribution of men and women according to the three salary categories: Worker, agent and manager.

	Man	Wife	Total
Worker	34.1 %	15.9 %	50 %
Agent	17.4 %	12.6 %	30 %
Manager	6.5 %	13.5 %	20 %
Total	58 %	42 %	100 %

A record of an employee is randomly selected and the following events are considered:

F: "the file chosen is that of a woman"
H: "The file chosen is that of a man"
O: "the file chosen is that of a worker"
A: "the file chosen is that of an agent"
C: "the file chosen is that of a manager"

Calculate the probabilities of events F, H, O, A, C, $C \cap F$, $O \cap F$, et $A \cap H$.

Solution:

P (F) = 0.42, P (H) = 0.58, P (O) = 0.5, P (A) = 0.3, P (C) = 0.2, $P(C \cap F) = 0.135$, $P(O \cap F) = 0.159$, $P(A \cap H) = 0.174$.

Example 2

A box contains five red balls numbered 0, 0, 1, 2 and 3 and four green balls numbered 0, 0, 2 and 3.

1. A test consists of drawing three balls simultaneously and at random from the box.
   Calculate the probability of each of the following events:
   A: "Get a single red ball"
   B: "Get a single ball with the number 0"
   C: "Get a single red ball with the number 0."
2. Two balls are drawn at random, successively without replacement.
   Calculate the probability of each of the following events:
   A: "Get a single ball with an even number"
   B: "The first ball drawn is red"

Solution:

1. $P(A) = P(RVV) = \dfrac{C_5^1 \times C_4^2}{C_9^3} = \dfrac{6 \times 5}{84} = \dfrac{30}{84} = \dfrac{5}{14}$. $P(B) = \dfrac{C_4^1 \times C_5^2}{C_9^3} = \dfrac{40}{84} = \dfrac{10}{21}$.
   $P(C) = \dfrac{C_2^1 \times C_7^2}{C_9^3} = \dfrac{2 \times 21}{84} = \dfrac{42}{84} = \dfrac{1}{2}$.
2. $P(A) = \dfrac{4}{9} \times \dfrac{5}{8} \times 2 = \dfrac{5}{9}$; $P(B) = \dfrac{5}{9} \times \dfrac{4}{8} = \dfrac{20}{72} = \dfrac{5}{18}$.

Example 3

An urn contains three white balls and five black balls.

1. Two balls are drawn at random, successively and without delivery.
   Calculate the probability of getting:
   a) A white ball then a black ball.
   b) A black ball and then a white ball.
   c) Two balls of different colors.
2. The test now consists of successively and randomly drawing two balls from the urn in the following way:

- If the ball drawn is white, it is put back in the urn, then a second ball is drawn.
- If the ball drawn is black, it is kept, then a second ball is drawn from the urn.
  Calculate the probability of getting:
  a) A white ball then a black ball.

b) A black ball and then a white ball.
c) Two balls of different colors.

<u>Solution:</u>

1. a) P (White then black) = $\frac{A_3^1 \times A_5^1}{A_8^2} = \frac{15}{56}$.

   b) P (Black then white) = $\frac{A_5^1 \times A_3^1}{A_8^2} = \frac{15}{56}$.

   c) P (Two balls of different colors) = $\frac{15}{56} + \frac{15}{56} = \frac{30}{56} = \frac{15}{28}$.

2. a) P (White then black) = $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$.

   b) P (Black then white) = $\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$.

   c) P (Two balls of different colors) = P (a) + P (b) = $\frac{15}{64} + \frac{15}{56} = \frac{225}{448}$.