CHAPTER 1: Complex Numbers

Some students have difficulty understanding the beginning of the lessons on complexes, this is normal because there is a break in the understanding of sets.

In fact, this novelty in the understanding of mathematics is not a first because in the previous classes they also studied some new things in the sets.

In the previous classes we saw the set N where we could solve the equations of type $x + 3 = 7$: The solution is $x = 4$. However, in this set the equations of type $x + 3 = 0$ have no solution, hence the idea of extending the set of numbers: in Z the solution is $x = -3$.

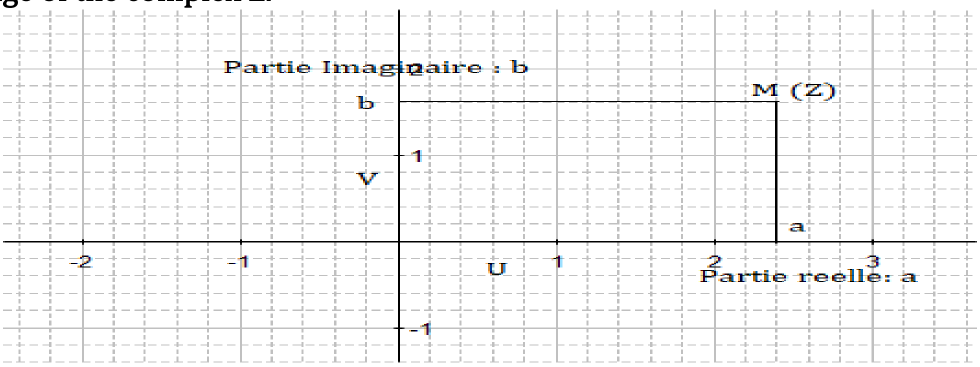
Similarly, the equations type $3x + 4 = 0$ require the use of a new set Q where the solution is $x = -4/3$. But equations of type $x^2 - 3 = 0$ had no solution, hence the use of the set R with the square root of a number, the solution was thus found $x = \sqrt{3}$ or $x = -\sqrt{3}$. However, equations of type $Z^2 + 1 = 0$ closer to the previous one had no solution in R, hence the use of complex numbers C in which there exists an imaginary i such that $i^2 = -1$:

The solutions of $Z^2 + 1 = 0$ are i and – i.

The quadratic equations with a negative Δ discriminant and many others are solved in C. It is this set of complex numbers that we will study in this chapter.

I) Algebraic form of a complex number:

1. <u>Definition</u>: Any number written in the form $a + ib$ with $a \in R$, $b \in R$ and i such that $i^2 = -1$ is in algebraic form.
   a is called the real part and denoted Re (Z) and b is the imaginary part denoted Im (Z).
   N.B: All real is also a complex whose imaginary part is zero, i.e.
   Im (Z) = 0. $R \subset C$ so $N \subset Z \subset Q \subset R \subset C$.
2. <u>Graphical representation of the image M of a complex $Z = a + ib$:</u>
   It is the point with coordinates M (a; b), Z is called the affix of the point M and M is said to be the image of the complex Z.



<u>Property:</u>
If Z and Z' are two complexes such that $Z = a + ib$ and $Z' = a' + ib'$.
We have: $Z = Z'$ if and only if $a = a'$ and $b = b'$: that is to say if the real parts and imaginary are equal to each other.

3. <u>Conjugate of a complex number</u>
   <u>Definition:</u> Let the complex $Z = a + ib$ be called the conjugate of Z the complex number $\overline{Z}$ such as: $\overline{Z} = a - ib$.
   <u>Geometric interpretation of the conjugate:</u>
   The images of a Z complex and its conjugate $\overline{Z}$ are symmetrical with respect to the x-axis or the real axis.
   <u>Examples:</u> The conjugates of 3 + 2 i; 5 i and 2 are 3 – 2 i; - 5 i and 2 respectively.
   <u>Properties:</u> For any complex number Z, such that $Z = a + ib$:
   P 1: $\overline{\overline{Z}} = Z$. P 2: $Z + \overline{Z} = 2\operatorname{Re}(Z)$ or $Z + \overline{Z} = 2a$.
   P 3: $Z \times \overline{Z} = [\operatorname{Re}(Z)]^2 + [\operatorname{Im}(Z)]^2$ that is to say: $Z \times \overline{Z} = a^2 + b^2$.
   P 4: $Z - \overline{Z} = 2\operatorname{Im}(Z)$ or $Z - \overline{Z} = 2ib$.

P 5: $\begin{cases} Z \text{ est reel si et seulement si } Z = \overline{Z} \\ Z \text{ est imaginaire pure si et seulement si } Z = -\overline{Z}. \end{cases}$

P 6: For all Z and Z' ∈ C.

(i); $\overline{Z+Z'} = \overline{Z} + \overline{Z'}$ (ii) $\overline{Z \times Z'} = \overline{Z} \times \overline{Z'}$; (iii) $\overline{\left(\frac{Z}{Z'}\right)} = \frac{\overline{Z}}{\overline{Z'}}$.

Rule: To write a quotient of complexes in algebraic form, we multiply the numerator and the denominator by the conjugate of the denominator.

Example: $Z = \frac{4+2i}{2-3i} = \frac{(4+2i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)} = \frac{8+12i+4i-6}{4+9} = \frac{2+16i}{13} = \frac{2}{13} + \frac{16}{13}i$.

We have : $\operatorname{Re}(Z) = \frac{2}{13}$ and Im (Z) = $\frac{16}{13}$.

4. Inverse and opposite of a complex:

a) Definition: The inverse of a non-zero Z complex is the $\frac{1}{Z}$.

Property: If Z = a + i b we have: $\frac{1}{Z} = \frac{\overline{Z}}{Z\overline{Z}} = \frac{\overline{Z}}{a^2+b^2} = \frac{a-ib}{a^2+b^2}$.

b) Opposite of a complex:

Let be a complex Z written in the form Z = x + i y the opposite of Z denoted – Z = - x – i y.

Property: Z + opposite of (Z) = 0.

Geometric interpretation of the opposite.

The images of a complex and its opposite are symmetrical with respect to the origin of the coordinate system.

5. Power of a complex number:

Definition: Z being a nonzero complex.

a- $Z^0 = 1$; b- $0^n = 0$; c- $Z^{n+1} = Z^n \times Z$; d- $Z^{-n} = \frac{1}{Z^n}$;

For all n ∈ N We have: a) $i^{4n} = 1$; b) $i^{4n+1} = i$; c) $i^{4n+2} = -1$; d) $i^{4n+3} = -i$.

Newton's binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + \ldots\ldots + C_n^n a^0 b^n.$$

Example: Expand $(2-i)^4$.

$$(2-i)^4 = C_4^0 2^4 (-i)^0 + C_4^1 (2)^3 (-i)^1 + C_4^2 (2)^2 (-i)^2 + C_4^3 (2)^1 (-i)^3 + C_4^4 (2)^0 (-i)^4 =$$

$(2-i)^4 = 16 - 32i - 24 + 8i + 1 = -7 - 16i$.

6. Module and argument of a complex number:

a- From the module

Definition:

Let the complex Z = a + i b.

We say that the real r = $\sqrt{a^2+b^2}$ is the modulus of Z and we denote $|Z| = r$: C, geometrically in the complex plane, $(O; \vec{u}; \vec{v})$ the distance that separates the image from the point M with the affix Z = a + i b at the origin O of the coordinate system.

Properties:

P 1: $|Z| = 0 \Leftrightarrow Z = 0$; P 2: $|Z| = |-Z| = |\overline{Z}| = |-\overline{Z}|$; P 3: $|Z| = \sqrt{Z\overline{Z}}$.

P 4: $\operatorname{Re}(Z) \le |Z|$ and $\operatorname{Im}(Z) \le |Z|$; P 5: $|Z+Z'| \le |Z| + |Z'|$ triangular inequality.

P 6: $|ZZ'| = |Z| \times |Z'|$;

For any complex Z ' ≠ 0 P 7: $\left|\frac{1}{Z'}\right| = \frac{1}{|Z'|}$; $P_8 : \left|\frac{Z}{Z'}\right| = \frac{|Z|}{|Z'|}$;

For any natural number n and m: $|Z^n| = |Z|^n$ and $|Z^n| \times |Z^m| = |Z|^{n+m}$.

b- The argument

Let the point M with affix Z = a + i b, the argument of Z is geometrically in the complex plane $(O; \vec{u}, \vec{v})$ the oriented angle $(\overrightarrow{Ou}; \overrightarrow{OM})$.

We note: Arg (Z) = $(\overrightarrow{Ou}; \overrightarrow{OM}) + 2k\pi$ with k ∈ Z.

Properties:

P 1: Z = Z ' if and only if $|Z| = |Z'|$ and arg (Z) = arg (Z') + 2k π; k ∈ Z.

P 2: arg (ZZ') = arg (Z) + arg (Z') + 2 k π; k ∈ Z.

For Z' non-zero: P 3: $\arg\left(\frac{1}{Z'}\right) = -\arg(Z') + 2k\pi;\ k \in \mathbb{Z}$, and

P 4: $\arg\left(\frac{Z}{Z'}\right) = \arg(Z) - \arg(Z') + 2k\pi$; k ∈ Z.

P 5: arg (Z^n) = n arg (Z) + 2 k π; k ∈ Z.

Calculating complex arguments:

1) Know the table of remarkable values.
2) Master the techniques for solving trigonometric equations cos (x) = α and sin x = β and that sin (- x) = - sin x; cos (- x) = cos (x) ....

Angle α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Kos (α)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sin (α)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

The other values can easily be deduced from the remarkable values.

Example: $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

II) Trigonometric form of a complex:

Definition: Let be the non-zero complex number Z, the trigonometric form of Z is its writing under the form Z = r (cos (θ) + i sin (θ)).

Property: For any non-zero complex number Z of modulus r and argument θ the writing of Z the form Z = r [Cos (θ) + i sin (θ)] is the trigonometric form of the complex Z.

Exercise: Write the following complexes in trigonometric form:

$Z_1 = 1 + i\sqrt{3};\ Z_2 = -1 + i\sqrt{3};\ Z_3 = -2 - 2i$ and $Z_4 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

Solution:

- $Z_1 = 1 + i\sqrt{3} = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$. car $|Z_1| = \sqrt{1+3} = \sqrt{4} = 2$ and arg (Z_1) = $\frac{\pi}{3} + 2k\pi$.
- $Z_2 = -1 + i\sqrt{3} = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$. $Z_3 = -2 - 2i = 2\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$.
- $Z_4 = \frac{\sqrt{3}}{2} + \frac{1}{2}i = 1.\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.

III) Exponential forms of a complex:

The third form of Z-complex notation is the exponential form.

Definition: Any nonzero complex number of modulus r and argument θ can be denoted $r e^{i\theta}$ with r ∈ R *, and θ ∈ R; e is a number and is read exponentially: e ≈ 2.718.

Example: Z = - 1 + i $\sqrt{3} = 2e^{\frac{2\pi i}{3}}$.

Properties: For any non-zero complex Z and Z' and θ ∈ R we have:

- If $Z = e^{i\theta}$ then $|Z| = r = 1$.
- $\left(\overline{re^{i\theta}}\right) = re^{-i\theta} = r \times \frac{1}{e^{i\theta}}$. If r = 1 then $\left(\overline{e^{i\theta}}\right) = e^{-i\theta} = \frac{1}{e^{i\theta}}$.
- If $Z = r\,e^{i\theta}$ and $Z' = r'\,e^{i\theta'}$ then $ZZ' = rr'e^{i(\theta+\theta')}$. $\frac{Z}{Z'} = \frac{r}{r'}e^{i(\theta-\theta')}$; $Z^n = r^n e^{in\theta}$.

IV) Other formulas and their applications to trigonometry:

1) Moivre's formula:

We have: cos (θ) + i sin (θ) $\Leftrightarrow (\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta}$ hence:

a- Moivre's formula:

For any real number θ and any relative integer n, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

b- Application:

To write cos (n θ) and sin (n θ) as a function of cos (θ) and sin (θ).

Example: $(\cos 4\theta + i\sin 4\theta) = ((\cos(\theta)) + i\sin(\theta))^4 =$

$\cos^4\theta + 4\cos^3\theta.(i\sin\theta) + 6\cos^2\theta.(i\sin\theta)^2 + 4\cos\theta.(isin\theta)^3 + (i\sin\theta)^4 =$

$[\cos^4\theta + \sin^4\theta - 6\cos^2\theta.\sin^2\theta] + i\left[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right]$

So cos (4 θ) = cos $^{4\theta}$ + sin $^{4\theta}$ – 6 cos $^{2\theta = 1 - 8\cos 2\theta + 8\cos 4}$ θ.

Et sin (4 θ) = 4cos θ. sin θ – 4 cos θ.sin ^{3} θ = 2 sin (2 θ) (1- 2sin ^{2} θ)

= 2 sin 2θ – 4 sin 2θ sin2θ.

2) Euler's formulas:

a- $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

b- Applications of the formulas:

These formulas are used to linearize trigonometric functions written in the form cos ^{n} (θ) and sin ^{n} (θ).

Example: Linearize cos ^{4}(θ):

$$\cos^4(\theta) = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{16}\left(e^{4i\theta} + 4e^{3i\theta}.e^{-i\theta} + 6e^{2i\theta}.e^{-2i\theta} + 4e^{i\theta}.e^{-3i\theta} + e^{-4i\theta}\right)$$

$$= \frac{1}{16}\left(e^{4i\theta} + e^{-4i\theta} + 4e^{2i\theta} + 4e^{-2i\theta} + 6\right) = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6) = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

Some formulas:

a- $1 - e^{i\theta} = e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right) = -2ie^{i\frac{\theta}{2}}.\sin\left(\frac{\theta}{2}\right)$.

b- $1 + e^{i\theta} = e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right) = 2e^{i\frac{\theta}{2}}.\cos\left(\frac{\theta}{2}\right)$.

V) Solving complex equations:

1) Operations in C:

a- Equations of type a Z ^{2} + b Z + c = 0; a, b and c are three real numbers:

Solve in C (E): x ^{2} + 1 = 0 and (F): x ^{2} + x + 1 = 0.

(E): x ^{2} + 1 = 0 gives x ^{2} = - 1 = i ^{2} implies x = + i or x = - i.

(F): x ^{2} + x + 1 = 0, Δ = 1 – 4 = - 3 < 0 therefore, Δ = 3 i ^{2}.

$x' = \frac{-1-\sqrt{3i^2}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $x'' = \frac{-1+\sqrt{3i^2}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. $S = \left\{\frac{-1-i\sqrt{3}}{2}; \frac{-1+i\sqrt{3}}{2}\right\}$.

b- Equation of type $\frac{a+biz}{cz} = z'$ and $(a+ib)z = \bar{z} + a' + ib'$ with a, b, a' and b' ∈ R.

i) **Resolve:** $\frac{1+2iz}{z}=1-2i \Leftrightarrow 1+2i=(1-2i)z \Leftrightarrow (1-2i)z-2iz=1 \Leftrightarrow z[1-2i-2i]=1$

$$z(1-4i)=1 \Leftrightarrow z=\frac{1}{1-4i}\times\frac{1+4i}{1+4i}=\frac{1}{17}+\frac{4}{17}i.$$

ii) Resolve: $(3-2i)z=\bar{z}-2-16i.$

Solution: Let us first set z = x + i y and $\bar{z}=x-iy.$

$(3-2i)z=\bar{z}-2-16i \Leftrightarrow (3-2i)(x+iy)=x-iy-2-16i \Leftrightarrow 3x+3iy-2ix+2y=x-iy-2-16i \Leftrightarrow$

$3x+2y-x+2+i(-2x+3y+y+16)=0 \Leftrightarrow \begin{cases}2x+2y+2=0\\-2x+4y+16=0\end{cases}$ Hence x = 2 and y = - 3. Consequently Z = 2 – 3i.

2) Nth root of a complex number

a) General case:

Definition: n is a non-zero natural number and Z is a complex number, we call the nth root of Z the complex z' such that (z') ^{n} = Z.

Look for the nth roots of Z = r (cos θ + i sin θ). Let's ask $z'=\rho(\cos\alpha+i\sin\alpha)$

$\Leftrightarrow (z')^n=\rho^n(\cos n\alpha+i\sin n\alpha)$ we have (z') ^{n} = Z $\Leftrightarrow \begin{cases}\rho^n=r \Leftrightarrow \rho=\sqrt[n]{r}\\ n\alpha=\theta+2k\pi;\ k\in\mathbb{Z} \Leftrightarrow \alpha=\frac{\theta}{n}+\frac{2k\pi}{n};\ k\in\mathbb{Z}.\end{cases}$

Property: Any non-zero complex Z = r (cos θ + i sin θ) admits n nth roots which are written under the

form: $z=\sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right]$ with k ∈ {0; 1 ; 2 ; 3 ; 4 ;.... ;(n-1)}

and n ∈ N\{0; 1}.

Geometric interpretation: In an orthonormal coordinate system $(O;\vec{u},\vec{v})$, the image points of the n – roots

nth are on the circle (C) of center O and radius $\sqrt[n]{r}$.

If n = 2, the image points of the 2 square roots are diametrically opposite on (C).

If n > 2, the image points of the n – roots nth are the vertices of a regular polygon of n sides inscribed in the circle (C).

Exercise: Determine the cubic roots of the unit Z = 1 = (cos 0 + i sin 0) so $|Z|=1$;

Let z' be the cube roots z ' = r (cos α + i sinα) such that (z') ^{3} = Z.

$$(z')^3=(r(\cos\alpha+i\sin\alpha))^3=r^3(\cos3\alpha+i\sin3\alpha)=1 \Leftrightarrow \begin{cases}r^3=1 \Leftrightarrow r=1\\ 3\alpha=0+2k\pi \Leftrightarrow \alpha=\frac{2k\pi}{3};\ k\in\{-1;0;1;\}\end{cases}$$

If k = 0 then z' 1= cos 0 + i sin 0 = 1; If k = 1 then $z_2'=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

If k = - 1 then $z_3'=\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i.$

b) The square roots of a complex:

Let Z = a + i b and z = x + i y. We call the square roots of the Z complex the z complexes such that: Z = z ^{2}. The solution is equivalent to setting (x + i y) ^{2} = a + i b.

$$\begin{cases}|z|^2=|Z| & (1)\\ (\mathrm{Re}(z))^2-(\mathrm{Im}(z))^2=\mathrm{Re}(Z) & (2)\\ 2\,\mathrm{Re}(z)\times\mathrm{Im}(z)=\mathrm{Im}(Z) & (3)\end{cases} \Leftrightarrow \begin{cases}x^2+y^2=\sqrt{a^2+b^2} & (1)\\ x^2-y^2=a & (2)\\ 2xy=b & (3)\end{cases}$$

Example: Determine the square roots of the complex: $1 + i\sqrt{3}$.

$$\begin{cases} x^2+y^2=\sqrt{1+3}=\sqrt{4}=2 & (1) \\ x^2-y^2=1 & (2) \\ 2xy=\sqrt{3} & (3) \end{cases} \Leftrightarrow \begin{cases} (1)+(2) \Leftrightarrow 2x^2=3 \Leftrightarrow x^2=\frac{3}{2} \Leftrightarrow x=\sqrt{\frac{3}{2}};\ x=-\sqrt{\frac{3}{2}}. \\ (1)-(2) \Leftrightarrow 2y^2=1 \Leftrightarrow y^2=\frac{1}{2} \Leftrightarrow y=\frac{\sqrt{2}}{2};\ y=-\frac{\sqrt{2}}{2}. \\ (3)\ \textit{indique que x et y sont de meme signe} \end{cases}$$

The 2 roots are $z_1=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i;\quad z_2=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i.$

c) Geometric places

Let M, A and B be the respective affix points z, a and b or more explicitly M (z); A (a) and B (b).

- $\frac{z-a}{z-b}\in\mathbb{R}$ real: M is on the line (AB) deprived of the point B.
- $\frac{z-a}{z-b}\in i\mathbb{R}^*$ pure imaginary: M is on the circle of diameter [AB] deprived of the points A and B.
- Set of points M such that: $|z-a|=r$ with r > 0 : M is on the circle with center A and radius r.
- $|z-a|=|z-b|$: M is on the mediator of the segment [AB].

  For the whole suite k ∈ Z.
- (a) $\arg\left(\frac{z-a}{z-b}\right)=0+2k\pi$: M is on the segment [AB]deprived of points A and B.

  b) $\arg\left(\frac{z-a}{z-b}\right)=k\pi$: M is on the line (AB) deprived of the point B.

* $\arg\left(\frac{z-a}{z-b}\right)=\frac{\pi}{2}+k\pi$ or $\arg\left(\frac{z-a}{z-b}\right)=-\frac{\pi}{2}+k\pi$:

M is on the circle of diameter [AB] deprived of the points A and B.

COMPLEX EXERCISES

Exercise 1

1. Determine the argument and modulus of the following complexes:

   $a-\ Z_1=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2};\quad b-\ Z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}i;\quad c-\ Z_3=2i;\quad d-\ Z_4=2-2i;$

   $e-\ Z_5=1-i\quad;\quad f-\ Z_6=(2-2i)(1-i);\quad g-Z_7=-2ie^{i\theta}\sin(2\theta)\cos(\alpha)$

   With θ and α ∈ [0; π/6].
2. In each of the cases, recognize the geometric transformation t, the complex writing of which is:

   $a-\ Z'=Z+1+2i;\quad b-\ Z'=e^{i\frac{\pi}{5}}Z;\quad c-\ Z'=Z-3i$

   $d-Z'=-iZ\quad;\quad e-\ Z'=\frac{\sqrt{2}}{2}(1-i)Z;\quad f-Z'=\frac{1}{2}(1+i)Z.$

Exercise 2

We give $Z=\frac{\sqrt{3}}{2}+\frac{1}{2}i.$

1. Write Z in the trigonometric form.
2. Derive the writing of Z in exponential form.
3. Calculate Z^{2016} and Z^{2017}.
4. Derive a comparison of Z^{2017} and Z.

Exercise 3

The plan is related to the $\left(O;\,\vec{u},\,\vec{v}\right)$ 1 cm unit marker. Consider the points A, B and C with respective affifixes: $Z_A = -1 + i\sqrt{3};\ \ Z_B = -1 - i\sqrt{3}$ and $Z_C = 2$.

1. a – Verify that $\dfrac{Z_B - Z_C}{Z_A - Z_C} = e^{i\frac{\pi}{3}}$.

   b- Deduce the nature of the triangle ABC.

   c- Determine the center and radius of the circle (C) circumscribed to the triangle ABC.
2. a – Establish that the set (C2) of the points M of affix Z that verify : $2\left(Z + \overline{Z}\right) + Z\overline{Z} = 0$ is a center circle I of affix - 2. Specify its radius.

   b- Check that points A and B are elements of (C_2).
3. We give the point D affix – 4.

   a- Interpret geometrically the modulus and the argument of $\dfrac{Z_B - Z_A}{Z_D - Z_C}$.

   b- From this, deduce the exact nature of the ACBD quadrilateral.

<u>Exercise 4</u>

Let be the complex number $Z = \dfrac{z}{z-1}$ where z is a complex number other than 1.

1. a – To $z = \dfrac{1}{2} + \dfrac{1}{2}i$, write Z in algebraic form.

   b- Show that if Z is real then z is also real.
2. We set Z = X + i Y and z = x + i y with x; y ; X and Y of the reals.

   Show that $X = \dfrac{x^2 + y^2 - x}{(x-1)^2 + y^2}$ and $Y = \dfrac{-y}{(x-1)^2 + y^2}$.
3. We relate to an orthonormal coordinate system the complex plane $\left(O;\,\vec{u},\,\vec{v}\right)$.

   a- Determine and construct the set (E) of points M with affix z such that Z is a real.

   b- Determine and construct the set (F) of the points M of affix z such that Z is pure imaginary.

<u>Exercise 5</u>

The plane is related to a direct orthonormal coordinate system $\left(O;\,\vec{u},\,\vec{v}\right)$. We give a graphic unit of 4 cm.

Let f be the function which, to any complex number z other than – 2 i, associates: $Z = f(z) = \dfrac{z - 2 + i}{z + 2i}$.

The respective affix points A and B are called $z_A = 2 - i$ and $z_B = -2i$.

1. Determine:

   a- The set E of points M with affix z such that Z is a real.

   b- The set F of the points M of affix z such that Z is a pure imaginary.

   c- The set G of the points M of affix z such that $|Z| = 1$.
2. Determine the sets E, F and G without using the real and imaginary parts of Z.
3. Represent these three together.
4. Calculate $|Z - 1| \times |z + 2i|$ and deduce that the points M_0 with affix Z. when the point M with affix z runs through the circle with center B and radius $\sqrt{5}$ are all on the same circle whose center affix and radius will be specified.

<u>Exercise 6</u>

The complex plane is related to a direct orthonormal coordinate system $\left(O;\,\vec{u},\,\vec{v}\right)$.

Let A, B and C be the respective affix points i, 1 + i and 1 – i.

We call f the application of the private plane of A in itself which, at any point M of affix z ($z \neq i$) associates the point M' of affix z' defined by: $z' = \dfrac{z^2}{1-z}$.

1. a – Place points A, B and C on a figure that will be completed as the exercise progresses.
   What can we say about points B and C?
   b- Determine the affixes of the points B' and C' images of B and C by the application f.
   Place these points.
2. Determine the affixes of the invariant points by f (the points M verifying f (M) = M).
3. Does the application f converse alignment? The environment?
4. We set z = x + i y and z' = x ' + i y ' with x, y, x' and y' reals.
   a- Demonstrate that: $x' = \dfrac{-x\left(x^2+y^2-2y\right)}{x^2+\left(1-y\right)^2}$.
   b- Derive the set E of the points M (z) such that z' is a pure imaginary.
   Represent the set E.

<u>Exercise 7</u>

In the complex plane is related to a direct orthonormal coordinate system $\left(O;\, \vec{u},\, \vec{v}\right)$.

We give the polynomial P (z) defined by any element z of C by:

$$P(z)=z^3-5(1+i)z^2-2(1-9i)z+16-8i.$$

1. a – Show that P (z) = 0 admits a pure imaginary root z_0 that we will calculate.
   b- Determine the polynomial q (z) such that we have P (z) = (z – z_0) q (z).
   Finish solving P (z) = 0, We will note z_1 and z_2 the other two solutions such that Im (z_1) < Im (z_2) where Im (z) denotes the imaginary part of z.
2. Let A, B and C be the respective affix points $z_A = z_1$; $z_B = z_0$ and $z_C = \overline{z_2}$.
   a- Calculate $\dfrac{z_C - z_A}{z_B - z_A}$.
   b- From this, deduce the nature of the triangle ABC.
3. Let t be the translation of vector $\overrightarrow{AB}$; M and M' the respective affix points z and z' such that M' is the image of the point M by the translation t.
   a- Express z' in terms of z.
   b- Calculate the affix of z_D image of C by the translation t.
   c- Give the exact nature of the quadrilateral ABDC.

<u>Exercise 8</u>

1. Consider the polynomial P: P (z) = $z^3+\left(1-i\sqrt{2}\right)z^2+\left(74-i\sqrt{2}\right)z-74i\sqrt{2}.$
   a- Determine the real number α such that α i is the solution of the equation P (z) = 0.
   b- Find two real numbers a and b such that for any complex number z, we have:
   P (z) = $\left(z-i\sqrt{2}\right)\left(z^2+az+b\right)$.
   c- Solve in the set C of complex numbers the solutions of the equation P (z) = 0.
2. The complex plane is related to a direct orthonormal coordinate system $\left(O;\, \vec{u},\, \vec{v}\right)$. Unit 1 cm.
   a- Place the points A, B and I with respective affixes z_A = - 7 + 5 i; z_B = - 7 – 5 i and z_I = $i\sqrt{2}$.
   b- Determine the affix of the image of the point I by the rotation of the center O and the angle $-\dfrac{\pi}{4}$.
   c- Place the point C with an affix z_C = 1 + i. Determine the affix of the point N such that ABCN is a parallelogram.
   d- Place the point D with affix z_D = 1 + 11 i. Calculate $Z = \dfrac{z_A - z_C}{z_D - z_B}$ in algebraic and trigonometric form.
   Prove that the lines (AC) and (BD) are perpendicular and deduce the nature of the quadrilateral ABCD.

BACCALAUREATE SUBJECTS

Typical topic 1

Exercise 1:

In a lottery, there are two sets of winning tickets: A and B.
Series A has 12 tickets including 4 winners, Series B has 15 tickets including 5 winners.
One person buys 3 tickets, 2 of which are from series A and 1 from series B.

1. What are the possible choices.
2. a – What is the probability that he will get a winning ticket?
   b- Or at the event: "obtain at least one winning ticket.
   Calculate P (A) in two distinct ways.
3. A winning ticket from Series A wins \$2500 and a winning ticket from Series B wins \$5000.
   X is the random variable associated with the purchase of 3 tickets (2 from series A and 1 from B)
   The gain realized:
   a- What is the set of values taken by X?
   b- Determine the probability distribution of X.

Exercise 2

The plane is related to a direct orthonormal coordinate system $\left(O;\vec{u},\vec{v}\right)$. We give a graphic unit of 4 cm.

Let f be the function which, to any complex number z other than – 2 i, associates: $Z=f\left(z\right)=\dfrac{z-2+i}{z+2i}$.

The respective affix points A and B are called $z_A = 2 - i$ and $z_B = -2i$.

5. Determine:
   d- The set E of points M with affix z such that Z is a real.
   e- The set F of the points M of affix z such that Z is a pure imaginary.
   f- The set G of the points M of affix z such that $|Z|=1$.
6. Determine the sets E, F and G without using the real and imaginary parts of Z.
7. Represent these three together.
8. Calculate $|Z-1|\times|z+2i|$ and deduce that the points M_0 with affix Z. when the point M with affix z runs through the circle with center B and radius $\sqrt{5}$ are all on the same circle whose center affix and radius will be specified.

Exercise 3

Let f be the function defined for any real x by: $f\left(x\right)=e^{2x}-\left(x+1\right)e^{x}$.

Part A:

Let the function h be two times differentiable defined on [0 ; + ∞ [by : $h\left(x\right)=e^{x}-\dfrac{x^2}{2}$.

1. a – Calculate h'(x) then h''(x).
   b- Determine the sign of h''(x) and prove that h' is increasing over [0; + ∞ [.
   c- Noting that h '(0) = 1 show that h is increasing over [0 ; + ∞ [and calculate h (0).
2. Deduce that for any positive or zero real x: $e^{x}-\dfrac{x^2}{2}\geq 1$.
3. Deduce $\lim\limits_{x\to+\infty}\dfrac{e^{x}}{x}$.

Part B:

Consider the function g defined on R by: g (x) = 2 e^{x} – x – 2.

1. Determine the limit of g in - ∞ then in + ∞.
2. Study the direction of variation of g and then draw up its table of variations.
3. a – Show that x = 0 is the solution of the equation g (x) = 0.
   b- Show that the equation g (x) = 0 admits of a second solution α over the interval] -1.6; -1,5 [
   c- Determine the sign of g on R.

Part C:

1. Determine the limit of f in - ∞ then in + ∞.
2. Calculate f '(x). Derive from part B the sign of f '(x) on R.
3. Show that: $f(\alpha) = -\dfrac{\alpha^2 + 2\alpha}{4}$ where α is defined in the previous part. From this deduce a frame of f (α).
4. Draw up a table of variations in f.
5. Draw the curve (C), representative of f in the plane relative to an orthonormal coordinate system
   (Graphic unit: 2 cm).

Subject Type 2

Exercise 1

In the set of complex numbers C: (E): $Z^3 - 3iZ^2 - (3-i)Z + 2 + 2i = 0$.

1. Show that (E) admits a real root Z_0 that we will determine.
2. To finish the resolution of (E), we recall that (E) admits 3 roots.
3. Let us consider Z_A, Z_B and Z_C the respective affixes of the points A, B and C such that Z_A is the real root and Z_B is the pure imaginary root.
   Place the points A, B and C in a 2 cm unit orthogonal coordinate system.
4. a – Calculate $Z = \dfrac{Z_C - Z_A}{Z_B - Z_A}$.
   b- Determine and interpret $|Z|$.
   c- Deduce the nature of the triangle ABC.

Exercise 2

1. Consider the differential equation (E_1): y ' + a y = b where a and b are two nonzero reals.
   We put $Z = y - \dfrac{b}{a}$.
   a- Determine a differential equation (E_2) satisfied by Z.
   b- Solve equation (E_2) and deduce the solutions of equation (E_1).
2. We denote by (E) the set of functions f differentiable on R that do not cancel on R, such that:
   $f'(x) + 2f(x) = [f(x)]^2$, for any real x.
   a- Let f be a solution function of (E), prove that the function defined by g (x) = $\dfrac{1}{f(x)}$ is differentiable on R and that it is the solution of a differential equation of the form y' + a y = b,
   (a and b real nonzero).
   b- From this, deduce that the solution functions of (E) are the functi~~ons~~ x $\dfrac{2}{2ke^{2x}+1}$ where k is a strictly positive real.

Exercise 3

We consider the function f defined from R^+ to R by: $f(x) = \begin{cases} x[1 - \ln(x)]^2 & pour \quad x > 0 \\ 0 & pour \quad x = 0 \end{cases}$

We denote by (C) its representative curve in an orthonormal coordinate system $(O; \vec{i}, \vec{j})$.

Part A:

1. Study the continuity of f in 0.

2. Study the differentiability of f to 0.
3. Study the variations of f and draw up a table of variations.
4. Build (C).

Part B:

1. Let α be a real number belonging to the interval $]0; e[$.
   Calculate using two integrations per part, the area A (α) of the part of the plane bounded by (OI), the lines of equations $x = \alpha$, $x = e$ and the curve (C).
2. Calculate the limit of A (α) when α approaches 0.
3. Determine the coordinates of the points of intersection of the curve (C) and the line of equation $y = x$.
4. For what value of α does the line (D_a) of the equation $y = a x$, intersect the curve (C) at two distinct or confused points M and M' other than O?

Part C:

1. Prove that the restriction g of the function f at the interval $[e; +\infty[$ admits a reciprocal application g^{-1} whose set of definitions will be specified.
2. Is a – g^{-1} differentiable in e?
   b- Calculate the image in e^2 by g^{-1}.
3. Construct the graphical representation of g^{-1} denoted (C').