Numerical Sequences:

Definition:

A numerical sequence is any definite function of IK ⊂ N or N with values in R, it can be defined by an explicit form.

Example: $U_n = 2n^2 + 3$ or by a recurrence form:

Example: $U_{n+1} = -U_n + 2$; $U_{n-1} = U_{n-2} + 7$ ....

We say that U_n is increasing if $U_{n+1} \geq U_n$ and decreasing if not; U_n is constant if and only if $U_{n+1} = U_n$. Whatever n belongs to a given interval I.

To find out:

The direction of variations of the sequence U_n

1st method: We make the difference $U_{n+1} - U_n$: and we analyze the sign:

- If $U_{n+1} - U_n \geq 0$ then U_n is increasing.
- If $U_{n+1} - U_n \leq 0$ then U_n is decreasing.
- If $U_{n+1} - U_n = 0$ then U_n is constant.

2 th method: We can also calculate $\frac{U_{n+1}}{U_n}$ and compare with 1.

If $\frac{U_{n+1}}{U_n} \geq 1$: U_n is increasing. If $\frac{U_{n+1}}{U_n} \leq 1$: U_n is decreasing. If $\frac{U_{n+1}}{U_n} = 1$: U_n is constant.

N.B: This second technique can only be used if the terms of the sequence U_n are all strictly positive at least from a certain rank. The 1st is therefore a general method.

I) Arithmetic sequences and geometric sequences

1) Arithmetic sequences:

a) Definition: A sequence U_n is said to be arithmetic if it is written in the form recurrence

$U_{n+1} = U_n + r$, where r is a real called the reason of the sequence U_n and a 1 st term which is at

Define: Generally we choose U_0 or U_1 as the 1st term.

b) Evidence:

To show that a sequence is arithmetic, we calculate $U_{n+1} - U_n$ if we find a real independent of n we conclude that U_n is arithmetic by taking care to specify the reason which is this found real and the 1 st term.

Example: $(U_n): \begin{cases} U_1 = 2 \\ U_{n+1} = -\frac{4}{3}n - 2 \end{cases}$ Show that (U_n) is arithmetic:

$U_{n+1} - U_n = -\frac{4}{3}(n+1) - 2 + \frac{4}{3}n + 2 = -\frac{4}{3} \in \mathbb{R}$ independent of n so the sequence U_n is arithmetic of reason $r = -\frac{4}{3}$ and of 1 st term $U_1 = 2$.

c) Expression of U_n as a function of n: Explicit form:

Let U_n be an arithmetic sequence of reason r and 1 st term U_k the explicit form of U_n is:

$U_n = U_k + (n - k)r$.

- If the 1st term is U_0 for example, then $U_n = U_0 + nr$.
- If the 1st term is U_1 then $U_n = U_1 + (n - 1)r$.

N.B: the reason of an arithmetic sequence r determines the direction of variations of U_n.

* If $r \leq 0$: U_n is decreasing.
* If $r \geq 0$: U_n is increasing.
* If $r = 0$: U_n is constant.

Example:

$\begin{cases} U_{n+1} = U_n - \frac{4}{3} \\ U_1 = 2 \ ; \ r = -\frac{4}{3} \end{cases}$; $U_n = U_1 + (n-1)r = 2 + (n-1)\left(-\frac{4}{3}\right) = -\frac{4}{3}n + \frac{10}{3}$.

d) Sum of the consecutive terms of an arithmetic sequence:

Let us calculate $S_n = U_0 + U_1 + U_2 + ... + U_n = \sum_{i=0}^{n} U_i$.

$$S_n = U_0 + (U_0 + r) + (U_0 + 2r) + ... + (U_0 + nr) = (n+1)U_0 + 1 + 2 + 3 + ... + n.$$

Now it is easy to establish by recurrence that: $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ which leads to

$$S_n = (n+1)U_0 + \frac{n(n+1)}{2} r = (n+1)\left(U_0 + \frac{nr}{2}\right) = \frac{(n+1)(U_0 + U_n)}{2}.$$

Note: We say that reals a, b and c follow each other in this order in an arithmetic progression if $b = \frac{a+c}{2}$.

Example: Say whether 1; 3 and 5 follow each other in this order in an arithmetic progression.

Same exercise for the following numbers:

$a)$ 2; -3 and – 8? (b) $\frac{1}{3}; \frac{7}{3}$ and $\frac{13}{3}$? (c) – 3; 2 and – 8? d) 2 i; 2 (i – e) and – 4 e + 2i.

2) Geometric sequences:

a) Definition:

The sequence V_n is qualified as geometric if $V_{n+1} = q V_n$.

The real q is called the reason of V_n, the 1st term of V_n is to be defined, we generally take V_0 or V_1.

b) Evidence:

To show that a sequence V_n is geometric, it is necessary and sufficient to show that the ratio $\frac{V_{n+1}}{V_n} \in \mathbb{R}$ and independent of n. If this ratio is a real independent of n, we conclude that V_n is geometric.

Example: Show that V_n is geometric, we will specify the reason and the 1st term.

$\begin{cases} V_0 = 2e \\ V_{n+1} = 2e^{n+1} \end{cases}$. We have: $\frac{V_{n+1}}{V_n} = \frac{2e^{n+1}}{2e^n} = e \approx 2,718 \in \mathbb{R}$ independent of n, so V_n is a sequence of reason q = e and the 1st term V_0 = 2nd.

c) Explicit form of a geometric sequence V_n as a function of n:

A geometric sequence (V_n) of reason q and 1 st term V_k is written: $V_n = V_k (q)^{n-k}$

If $V_k = V_0$ then, $V_n = V_0 q^n$. If $V_k = V_1$ then, $V_n = V_1 q^{n-1}$.

d) Sum of the terms of a geometric sequence:

$$S_n = V_0 + V_1 + V_2 + ... + V_n = \sum_{i=0}^{n} V_i = V_0 + V_0 q + V_0 q^2 + ... + V_0 q^n = V_0 + V_0\left(q + q^2 + ... + q^n\right).$$

And $q S_n = q (V0 + V0 (q + q^2 + q^3 + ... + q^n))$.

So: $S_n - q S_n = V_0(1-q) + V_0\left(q - q^2\right) + V_0\left(q^2 - q^3\right) + ...V_0\left(q^n - q^{n+1}\right)$

$$S_n(1-q) = V_0\left(1 - q + q - q^2 + q^2 - q^3 + q^3 + ... + q^n - q^{n+1}\right) = V_0\left(1 - q^{n+1}\right).$$

As a result: $S_n = V_0\left[\frac{1-q^{n+1}}{1-q}\right]$.

N.B: We can use the reason q to deduce the direction of variations of a geometric sequence.

If all the terms of the sequence are positive, then

- If $q \geq 1$ then, V_n is increasing.
- If $q \leq 1$ then, V_n is decreasing.
- If $q = 1$ then, V_n is constant.

II) Analysis of the convergence of a sequence:

1) Convergent and divergent sequence:

▶ A sequence (U_n) is convergent if it admits a finite limit in addition to infinity: If $\lim_{n\to+\infty} U_n = L$

With L is real, we say that (U_n) converges to the real L.

Example: $U_n = e^{\left(1+\frac{1}{n}\right)}$; $\lim\limits_{n\to+\infty} U_n = e$ because $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)=1$. ($U_n$) is said to converge to e.

▶ A sequence is said to be divergent if it does not admit a finite limit in infinity: Either the limit does not exist, just as the sequence is divergent.

2) Enhanced and reduced suite:

● A sequence U_n is increased by the real M if for all n we have: $U_n \leq M$. We say that M is major of the sequence.

Property: An increased and increasing sequence is necessarily convergent.

● A sequence U_n is reduced by the real m if for all n we have: $U_n \geq m$. We say that m is the minorant of the sequence.

Property: A diminished and decreasing sequence is necessarily convergent.

To show that a sequence (U_n) is increased, we study the sign of $U_n - M$.

If $U_n - M \leq 0$, we conclude that U_n is increased by M.

Similarly, if $U_n - m \geq 0$, we conclude that U_n is reduced by m.

3) Limit theorems:

a) Limitations and operations:

All properties on the limit of a sum, a product, and the limit of a quotient of functions are the same for sequences. The difference is that the limits of the sequences are calculated only in the vicinity of $+\infty$.

Cases of indeterminacy are dealt with on a case-by-case basis:

Use of conjugated expression; postmanship; The dominance rule expo dominates the power of x, which in turn dominates ln...........

Property: U_n and V_n on convergent sequences if from a certain index we have:

$U_n \leq V_n$ then $\lim\limits_{n\to+\infty} U_n \leq \lim\limits_{n\to+\infty} V_n$:this property allows us to compare the limits sequels.

b) Theorem: L is a real number, U_n; V_n and W_n of numerical sequences.

Divergent sequences:

▶ If from a certain index we have: $U_n \geq V_n$ and then $\lim\limits_{n\to+\infty} V_n = +\infty$ $\lim\limits_{n\to+\infty} U_n = +\infty$.

▶ If from a certain index we have: $U_n \leq V_n$ and then $\lim\limits_{n\to+\infty} V_n = -\infty$ $\lim\limits_{n\to+\infty} U_n = -\infty$.

▶ If from a certain index we have: $V_n \leq U_n \leq W_n$ and then $\lim\limits_{n\to+\infty} V_n = \lim\limits_{n\to+\infty} W_n = L$ $\lim\limits_{n\to+\infty} U_n = L$.

Exercise: studying the convergence of the suite $U_n = \dfrac{n^2+(-1)^n}{n^2}$.

Solution: Let n be a natural number, we can enclose the sequence U_n using two sequences convergent:

$\dfrac{n^2-1}{n^2} \leq \dfrac{n^2+(-1)^n}{n^2} \leq \dfrac{n^2+1}{n^2}$ or $\lim\limits_{n\to+\infty}\left(\dfrac{n^2-1}{n^2}\right) = \lim\limits_{n\to+\infty}\left(\dfrac{n^2+1}{n^2}\right)=1$, So, $\lim\limits_{n\to+\infty} U_n = 1$. The sequence U_n converges to 1.

Exercise: Studying the convergence of $U_n = \dfrac{-1}{n}\sin(n)$; $V_n = e^{\cos(n)}$ and $W_n = n - \ln(n)$.

Solution: We have: for all n ∈ N: $-1 \leq \sin(n) \leq 1$

$\Leftrightarrow \dfrac{-1}{n}\times(1) \leq \dfrac{-1}{n}\sin(n) \leq \dfrac{-1}{n}\times(-1) \Leftrightarrow \dfrac{-1}{n} \leq U_n \leq \dfrac{1}{n}$ or $\lim\limits_{n\to+\infty}\dfrac{1}{n} = \lim\limits_{n\to+\infty}\dfrac{-1}{n} = 0$ then $\lim\limits_{n\to+\infty} U_n = 0$.

We have: $-1 \leq \cos(n) \leq 1 \Leftrightarrow e^{-1} \leq e^{\cos(n)} \leq e^{1}$.

Limits of sequences of general term q^n:

▶ If $|q|<1 \Leftrightarrow -1<q<1$ then, $\lim\limits_{n\to+\infty} q^n = 0$: q^n converges to 0.

▶ If $q \leq -1$ then, q^n does not admit a limit: q^n diverges.

Let us calculate the limits of the following sequences $U_n = \left(\dfrac{-1}{3}\right)^n$, $V_n = \left(\dfrac{3}{2}\right)^{n+1}$ and $W_n = (-2)^n$.

■ $U_n = \left(\frac{-1}{3}\right)^n$. U n and in the form q n and q = $\frac{-1}{3}$ therefore, - 1 < q < 1 then $\lim_{n\to+\infty}\left(\frac{-1}{3}\right)^n = 0$.

■ $V_n = \left(\frac{3}{2}\right)^n$. V n and in the form q n and q = $\frac{3}{2}$ therefore, q > 1 then, $\lim_{n\to+\infty}\left(\frac{3}{2}\right)^n = +\infty$.

■ W n = (- 2) n. W n and in the form q n and q = - 2 < - 1 then $\lim_{n\to+\infty}(-2)^n$ does not exist.

4) Convergence of sequence of type $U_{n+1} = f(U_n)$:
Let f be a continuous function over an interval I = [a; b].
U_n is a sequence with values in I ($U_n \in I$) defined by the recurrence formula $U_{n+1} = f(U_n)$.
If U_n is convergent, then its limit is a α solution of the equation f (L) = L.

III) Proof by recurrence:

To show that a proposition P_n is true for all $n \geq n_0$ using the recurrence:

1) We verify that the proposition is true at the initial step n_0 (initial rank) possibly at the next rank.
2) We assume that the proposition is true up to order n and translate what this means, then show that the verification of the proposition at order n also leads to its verification at order n +1.
3) We conclude that the proposition is true at the order n, i.e. for all $n \geq n_0$.

Exercise 1

Let (U_n) : $\begin{cases} U_0 = 1 \\ U_{n+1} = \frac{1}{2}U_n + 2 \end{cases}$.

1. Represent the 1 st terms of the sequence.
2. Show by recurrence that for any n ∈ N : $0 \leq U_n \leq 4$.
   Deduce that (U_n) is increased.
3. To show that it is growing and to conclude as to its convergence.
4. Determine the limit of U_n.

Exercise 2

Let (V_n) be the sequence defined on N * by $V_1 = \frac{1}{2}$ and $V_{n+1} = \left(\frac{n+1}{2n}\right)V_n$.

1. Calculate V_2, V_3 and V_4.
2. a – Prove that for any integer n > 0, we have $V_n > 0$.
   b- Prove that the sequence ($V_{,n}$) is decreasing.
   c- Deduce that the sequence (V_n) is convergent.
3. Let (U_n) be the sequence defined by: $U_n = \frac{V_n}{n}$, (n > 0).
   a- Show that the sequence (U_n) is geometric. Specify the reason and the 1 st term.
   b- Express U_n and then V_n as a function of n.
   c- Calculate $\lim_{n\to+\infty} U_n$ and $\lim_{n\to+\infty} V_n$.

Exercise 3

We consider the sequence U_n defined on N by: $\begin{cases} U_0 = 1 \\ U_{n+1} = 4 - e^{-U_n}, \end{cases}$ For all n ∈ N.

1. Show by recurrence that for any nonzero natural number n, $3 < U_n < 4$.
2. a – Show that for any nonzero natural number n,
   $U_{n+1} - U_n$ and $U_n - U_{n+1}$ are of the same sign.
   b- Deduce the direction of variations of the sequence (U_n).
3. Study the convergence of the sequence (U_n).

Exercise 4

We consider the sequence of general terms (U_n) defined by: $U_n = \frac{-1+3U_{n-1}}{1+U_{n-1}}$; ($n \geq 3$) and for its 1st term $U_2 = 3$.

1. Calculate U_3, U_4 and deduce that the sequence (U_n) is neither arithmetic nor geometric.
2. a – Show that (U_n) is reduced by 1.
   b- Show that (U_n) is decreasing.
   c- Deduce that (U_n) is convergent and determine its limit.
3. We put $V_n = \frac{1+U_n}{-1+U_n}$.
   a- Show that (V_n) is an arithmetic sequence for which we will determine the 1st term and the reason.
   b- Express V_n and then U_n as a function of n.

Exercise 5

We consider the numerical sequence (U_n) defined on N by: $\begin{cases} U_0 = \frac{1}{2} \\ U_{n+1} = \frac{3U_n}{1+2U_n} \end{cases}$.

We assume that $U_n > 0$, for any natural number n.

1. Prove by recurrence that for any natural number n, $U_n < 1$.
2. Demonstrate that (U_n) is increasing and convergent.
3. We set $V_n = \frac{U_n}{1-U_n}$ for all $n \in N$.
   a- Prove that (V_n) is a geometric sequence whose reason and first term will be specified.
   b- Express V_n in terms of n. From this it can be deduced that $U_n = \frac{3^n}{1+3^n}$.
   c- Calculate the limit of (U_n).